Electromagnetic Fields in 2+1 Dimensions

by

Jonathan Zev Simon '85

THESIS

Submitted to
Princeton University
Department of Physics

In Partial Fulfillment of the Requirements for the Degree of

Bachelor of Arts

April 19, 1985

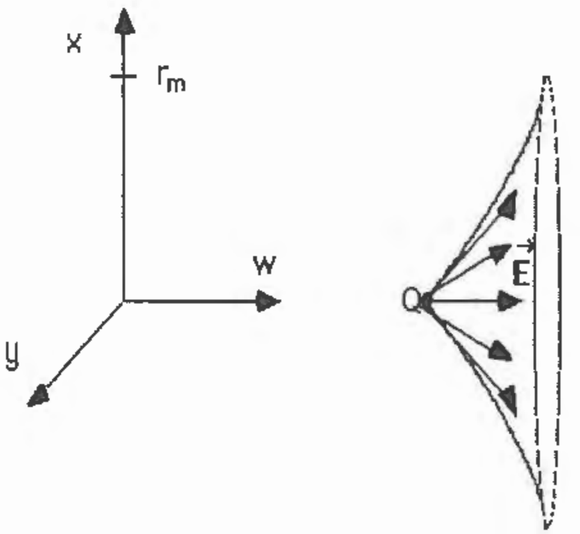
This thesis represents my own work in accordance with Princeton University regulations and academic ethics.

Jonathan Z. Simon

Errata

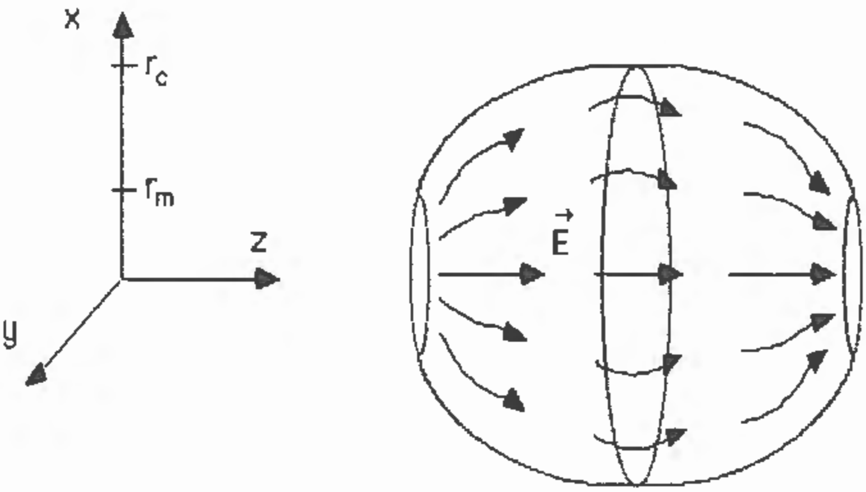
Page	Location	Correction
1	4th Line	Change last $R^{\sigma}{}_{\nu\alpha}$ to $\Gamma^{\sigma}{}_{\nu\alpha}$
2	5th Line	Change 4 to 3
21	1st Line	Change + $g_{\beta\gamma,\alpha}$ to - $g_{\beta\gamma,\alpha}$
34	4th Equation	Change V(r) to -V(r)
6	4th Eq	Change $r^2 d\phi^2$ to $r'^2 d\phi'^2$

30 Diagram 4.1 should be:

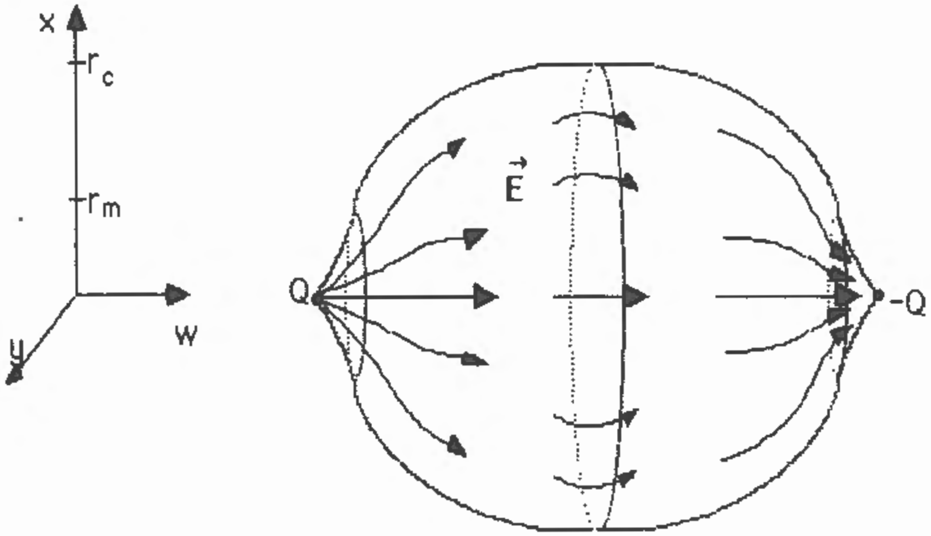


---

p. 31 Diagram 4.2 should be:



p. 31	Diagram 4.3	and
p. 46	Diagram 5.2	should be:



Dedication

To my mother, Nancy Simon. I owe you so much, and I miss you so much.

Table of Contents

Note on Conventions..1

Preface..2

Chapter 1
The Physics of Flatland..4

Chapter 2
Kaluza-Klein Theory..13

Chapter 3
Kaluza-Klein Wave...17

Chapter 4
Stationary Charge..29

Chapter 5
Inside-out "Black Hole" in Flatland...........................38

Appendix A..49

Appendix B..50

Bibliography..52

Notes on Conventions

In this paper I will be using the conventions used by Misner, Thorne, and Wheeler's Gravitation. They are as follows:

$g_{\mu\nu}$ has signature $(-+++)$

$R^{\mu}{}_{\nu\alpha\beta} = \Gamma^{\mu}{}_{\nu\beta,\alpha} - \Gamma^{\mu}{}_{\nu\alpha,\beta} + \Gamma^{\mu}{}_{\sigma\alpha}\Gamma^{\sigma}{}_{\nu\beta} - \Gamma^{\mu}{}_{\sigma\beta}R^{\sigma}{}_{\nu\alpha}$

$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$

$G_{\mu\nu} = R_{\mu\nu} - {}^{1}/_{2}g_{\mu\nu}R = \kappa T_{\mu\nu}$

$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$

The other important convention of which the reader should be aware concerns subscript labeling. As usual, Greek indices (α,β,μ,ν,etc.) run from 0 to 3. Latin indices (a,b,c,i,j, etc.), however, run from 0 to 2, not from 1 to 3.

Of course the speed of light is set to unity ($c=1$), and the Einstein summation convention is in use, even with Latin indices.

Preface

This thesis was inspired by and is based upon work done by my advisor, Professor Richard Gott, and his former student, Mark Alpert. Their paper was the first of three papers, all published within several months of each other, and all independently announcing the same result: General Relativity in 4 dimensions, 1 less dimension than ours, is not trivial. The goal of this thesis is to explore one facet of this 3 dimensional universe (first named "Flatland" by the Victorian novelist Edwin Abbot) --the workings of electromagnetism.

The first example presented is the case of an electromagnetic wave. In the case of 3 dimensions, the Kaluza-Klein formalism proves to be an especially alluring tool to study this example with. The goal of a Kaluza-Klein theory is to unify gravitational interactions with other interactions using General Relativity and additional microscopic dimensions. Unfortunately, the addition of extra dimensions makes the theory harder to visualize. "In what direction does the extra dimension point?" the layman might ask. Adding an extra dimension to Flatland, however, makes the new universe very similar to ours, thereby making the Physics even easier to understand. For instance, there are no gravitational waves in 3 dimensions (see Chapter 1), but there should be in 4 dimensions. By replacing electromagnetism with an extra dimension, an electromagnetic wave should be replaced by a gravitational wave.

The second case this thesis explores is that of an electrostatic charge. It turns out that this case is not as easily understood or solved using the Kaluza-Klein formalism (which is, unfortunately, at cross purposes with my original intentions). The case does, however, prove a very interesting cosmology for Flatland, which will be discussed in Chapter 5.

Special and much deserved thanks go to my advisor, Professor Richard Gott, for the tremendous amount of time and the wealth of ideas he has given me. Without his generous help this thesis would have been nothing.

Chapter 1

THE PHYSICS OF FLATLAND

On the surface, General Relativity would seem to have the same properties in 3 dimensions as in 4 dimensions. We still use a metric g_{mn} (equivalent to $g_{\mu\nu}$), equal to the upper lefthand 3×3 submatrix of $g_{\mu\nu}$, and from g_{mn} are derived the connection coefficients $\Gamma^{m}{}_{ab}$ (equivalent to $\Gamma^{\mu}{}_{\alpha\beta}$), the Riemann curvature tensor, R_{abcd} (equivalent to $R_{\alpha\beta\gamma\delta}$), and so on for all the other important quantities used in 4 dimensions. The most important equation of Einstein's General Relativity also still holds: $G_{mn}=(R_{mn}-{}^{1}/{}_{2}g_{mn}R)=\kappa T_{mn}$. In 4 dimensions, the constant κ is determined by the necessity that Einstein's equations reduce to Newton's equations in the non-relativistic limit, which forces $\kappa=8\pi G$ (G is Newton's Gravitational Constant). In 3 dimensions, as shall be shown, there is no Newtonian limit, and so κ remains an arbitrary constant. We note that in geometrized units, where κ is dimensionless (and $c = 1$), that $G_{\mu\nu}$ has units of curvature or $(\text{length})^{-2}$, and $T_{\mu\nu}$ has units of mass density or $(\text{mass})(\text{length})^{-2}$, which makes mass dimensionless as well.

The most important difference between the two cases is the different numbers of components in a tensor of a given rank. The number of components of a tensor comes from the dimension of the space raised to the power of the order of the tensor. The number of actually independent components of the tensor is determined by various symmetries of the tensor. Thus, while a vector changes from 4 components to 3 components, a tensor of rank 4 changes from 256 components to a mere 81 components. The tensor $g_{\mu\nu}$, with 10 independent components (the metric must be symmetric), becomes g_{mn} with only 6 independent components. $R_{\mu\nu}$ also goes from 10 to 6 components when it changes to R_{mn}. The tensor $R_{\alpha\beta\gamma\delta}$ with 20 independent components (due to various symmetries and antisymmetries), becomes R_{abcd} with a mere 6 independent components. The fact that R_{mn} and R_{abcd} have the same number of independent components makes it very plausible that R_{abcd} could be given from R_{mn} alone. This is indeed true, as shown in the formula:

$$R_{abcd} = g_{ac}R_{bd} - g_{ad}R_{bc} - g_{bc}R_{ad} + g_{bd}R_{ac} - {}^1/_2(g_{ac}\, g_{bd} - g_{ad}\, g_{bc})\, g^{mn} R_{mn}.$$

This becomes especially important in the absence of mass, where $T_{mn}=0$. From Einstein's equation $R_{mn}=0$ also, and therefore $R_{abcd}=0$ as well. This procludes any curvature at all in the absence of matter, whether in the form of gravitational waves or attraction at a distance. This is obviously different from the 4 dimensional case, where the Schwarzchild solution generates curvature outside the radius of a massive body. This required flatness of space-time in the absence of mass would seem to make the physics of gravity in 3 dimensions almost trivial.

It has been shown, however, by Gott and Alpert[1] (and independently by others[2]) that, although the presence of mass cannot induce curvature *per se* at a distance, it does affect the space around the mass by reducing the total amount of angle of the space by an amount proportional to the mass (from 2π to $2\pi-\kappa M$). This makes the space technically flat, but conical in form. Gott and Alpert give the metric around a point mass as:

$$ds^2 = -dt^2 + \left(\frac{2\pi}{2\pi-\kappa M}\right)^2 dr^2 + r^2 d\varphi^2.$$

This metric is analogous to the Schwarzchild metric around a point mass in 4 dimensions. The Gott-Alpert metric, however, does not lead to a black hole, and it can easily be seen that this metric corresponds to flat space-time by the simple transformations:

$$r' = \left(\frac{2\pi}{2\pi-\kappa M}\right) r$$

$$\varphi' = \left(\frac{2\pi-\kappa M}{2\pi}\right) \varphi$$

This gives the obviously flat metric:

$$ds^2 = -dt^2 + dr'^2 + r^2 d\varphi'^2.$$

We must be careful to notice that the limits of φ' are now different from the limits of φ: $0 \leq \varphi' \leq 2\pi-\kappa M$ rather than $0 \leq \varphi \leq 2\pi$. There is an angle deficit of κM induced by the presence of mass. This has the effect of causing initially parallel rays to converge as they pass on different sides of the particle, even though there is no curvature or attraction. This type of singularity is called quasi-regular, having the property that as $r \to 0$ the Riemann curvature is bounded, but the circumference is not equal to $2\pi r$.

Another solution that Gott and Alpert give for Einstein's equation in 3 dimensions is a static, dust-filled universe of radius r_0. The metric is:

$$ds^2 = -dt^2 + r_0^2(d\theta^2 + \sin^2\theta \, d\varphi^2)$$

This is a universe with energy density $T^{00} = {}^1/_{\kappa r_0{}^2}$ but zero pressure (since there is no attraction between the particles). The total area of the universe is $4\pi r_0^2$, giving total mass ${}^{4\pi}/_{\kappa}$ independent of the size of the universe (recall that mass is dimensionless).

The physics of electromagnetism also becomes simplified in three dimensions. In 4 dimensions, both the electric and magnetic fields are described by the single anti-symmetric tensor $F^{\mu\nu}$:

$$F^{\mu\nu} = \begin{bmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{bmatrix}.$$

This tensor obviously gives 3 components of electric Field and 3 components of magnetic field, which are the vectors **E** and **B** respectively. To make the transformation to 3 dimensions, use the same method as to make g_{mn}, that is to use only the upper lefthand 3×3 submatrix of $F^{\mu\nu}$, setting it equal to F^{mn}:

$$F^{mn} = \begin{bmatrix} 0 & E_1 & E_2 \\ -E_1 & 0 & B \\ -E_2 & -B & 0 \end{bmatrix}.$$

Now the electric field $\mathbf{E} = (E_1, E_2)$ is still a vector, but the magnetic field B is a scalar (!). It is easily seen that by extrapolating the metric into 5 dimensions the magnetic field would become a tensor with 6 independent components. Although the electric field **E** is always a vector, it is only in 4 dimensions that the magnetic field happens to be a vector as well.

In 4 dimensions Maxwell's Equations can be stated as two tensor equations for $F^{\alpha\beta}$:

$$F_{\alpha\beta;\gamma} + F_{\gamma\alpha;\beta} + F_{\beta\gamma;\alpha} = 0 \qquad (1.1)$$

$$F^{\alpha\beta}{}_{;\beta} = kJ^{\alpha} \quad \text{where } k=4\pi \qquad (1.2)$$

The analog of equation (1.1) in 3 dimensions would naturally be:

$$F_{ab:c} + F_{ca:b} + F_{bc:a} = 0$$

The only non-trivial solution to this equation arises from the case that all indices are different, which gives:

$$0 = -\frac{\partial E_1}{\partial y} + \frac{\partial E_2}{\partial x} + \frac{\partial B}{\partial t} = \mathrm{curl}\,(\mathbf{E}) + \frac{\partial B}{\partial t}. \qquad (1.3)$$

This is obviously analogous to the 4 dimensional vector equation:

$$0 = \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t},$$

except that with only 2 spatial dimensions the curl of a vector is a scalar, which gives only 1 equation as opposed to the 3 equations implied by the vector equation (1.3). It should also be noted that $\nabla\cdot\mathbf{B} = 0$, which is the other equation that arises from equation (1.1) in 4 dimensions, has no analogy in Flatland where B is a scalar. From equation (1.2), we should get the analogous equation:

$$F^{ab}{}_{;b} = kJ^a. \qquad (1.4)$$

But there is no reason to assume that $k=4\pi$, since all quantities dependent on dimensionality should be suspect, and 4π is the amount of solid angle in 3 spatial dimensions. A more logical guess might be $k=2\pi$, the amount of angle in a plane. Let us examine equation (1.4) for the case in which a=0. This reduces to Gauss' Law:

$$\frac{\partial E_1}{\partial x} + \frac{\partial E_2}{\partial y} = \nabla\cdot E = k\rho,$$

where ρ is the (surface) charge density. Just as Gauss' law gives an r^{-2} dependency for the electric field of a stationary point charge in 3 spatial dimensions, there should be an r^{-1} dependency in 2 spatial dimensions. (This is because field strength should vary as the density of lines of force, which are r^{-2} and r^{-1} in 2 and 3 spatial dimensions respectively.) Rewriting Gauss' Law in integral form:

$$\oint \mathbf{E}\cdot\hat{\mathbf{n}}\, d\ell = kQ,$$

and inserting $E = Q/_r$:

$$kQ = \oint \mathbf{E}\cdot\hat{\mathbf{n}}\, dl = \oint_0^{2\pi} \frac{Q}{r} r d\theta = 2\pi Q \qquad \Rightarrow k=2\pi$$

Thus the 3 dimensional counterpart of equation (1.2) can be rewritten as

$$F^{ab}{}_{;b} = 2\pi J^a \qquad (1.5)$$

Actually, changing of k from 4π to 2π arises only from our decision to use Gaussian units. In either case the factor of k could have been absorbed into the definition of the unit of charge, as is the case with mksa units. It is also a common convention to set k=1 in both cases, absorbing dimensional factors into the charge, but we will continue to use Gaussian units.

Setting a=1,2 in equation (1.5), results in:

$$\frac{\partial B}{\partial y} - \frac{\partial E_1}{\partial t} = 2\pi J_1$$
$$\frac{\partial B}{\partial x} - \frac{\partial E_2}{\partial t} = 2\pi J_2 \qquad (1.6)$$

Which are analogous to Maxwell's equation in 4 dimensions:

$$\nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = 2\pi \mathbf{J}$$

Apparently there is some sort of analogy to the curl of B, even when B is a scalar. Equation (1.6) can be expressed in a more compact equation if we introduce a new operator, R, which acts on a vector by rotating it 90° in the clockwise direction:

$$R\mathbf{F} = R\,(F_1\hat{\mathbf{x}} + F_2\hat{\mathbf{y}}) = F_2\hat{\mathbf{x}} - F_1\hat{\mathbf{y}}$$

$$R\nabla = R\,(\frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\hat{\mathbf{y}}) = \frac{\partial}{\partial y}\hat{\mathbf{x}} - \frac{\partial}{\partial x}\hat{\mathbf{y}}$$

$$\nabla\cdot(R\mathbf{F}) = (\frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\hat{\mathbf{y}})\,.\,(F_2\hat{\mathbf{x}} - F_1\hat{\mathbf{y}}) = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \mathrm{curl}\,(\mathbf{F})$$

$$RR\mathbf{F} = R^2\mathbf{F} = -\mathbf{F}$$

$$(R\nabla)A = R(\nabla A).$$

This notation lets us express Maxwell's equations in Flatland as:

$$\nabla\cdot\mathbf{E} = 2\pi\rho$$

$$R\nabla B - \frac{\partial E}{\partial t} = 2\pi \mathbf{J}$$

$$\nabla\cdot(RE) + \frac{\partial B}{\partial t} = 0$$

One other important set of equations involving the electromagnetic tensor is its relation to the stress-energy tensor. In 4 dimensions:

$$T^{\mu\nu} = {}^{1}/_{4\pi}\,(g_{\alpha\beta}F^{\mu\alpha}F^{\nu\beta} - {}^{1}/_{4}\,g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}) \qquad (1.7)$$

In 3 dimensions this becomes:

$$T^{mn} = {}^{1}/_{2\pi}\,(g_{ab}F^{ma}F^{nb} - {}^{1}/_{4}\,g^{mn}F_{ab}F^{ab}) \qquad (1.8)$$

where, again the change from 4π to 2π comes from the use of Gaussian units. That the factor of $^{1}/_{4}$ remains the same, regardless of the number of dimensions, is very important. For a proof of equation (1.8) see Alpert[3], or just follow a derivation of equation (1.8) using the least action principle from any textbook, but integrate over dx^2dt instead of dx^3dt.

One result of electromagnetism in 3 dimensions that might seem curious at first is that R, the Ricci curvature scalar, if caused only by electromagnetic fields, is not always identically zero. This is different from the case in 4 dimensions, where it always vanishes if caused solely by $F^{\mu\nu}$. In fact 4 dimensions is the only case for which R vanishes automatically, as can be demonstrated:

In four dimensions:

$$T^{\alpha}{}_{\beta} = {}^{1}/_{4\pi}\,(F^{\alpha}{}_{\delta}F_{\beta}{}^{\delta} - {}^{1}/_{4}\,\delta^{\alpha}{}_{\beta}F_{\gamma\delta}F^{\gamma\delta})$$

$$T^{\alpha}{}_{\alpha} = {}^{1}/_{4\pi}\,(F^{\alpha}{}_{\delta}F_{\alpha}{}^{\delta} - {}^{1}/_{4}\,\delta^{\alpha}{}_{\alpha}F_{\gamma\delta}F^{\gamma\delta})$$

$$
\begin{aligned}
T &= T^{\alpha}{}_{\alpha} = {}^{1}/_{4\pi}\,(g^{\alpha\gamma}g_{\alpha\mu}F_{\gamma\delta}F^{\mu\delta} - {}^{1}/_{4}\,\delta^{\alpha}{}_{\alpha}F_{\gamma\delta}F^{\gamma\delta}) \\
&= {}^{1}/_{4\pi}\,(\delta^{\gamma}{}_{\mu}F_{\gamma\delta}F^{\mu\delta} - {}^{1}/_{4}\,\delta^{\alpha}{}_{\alpha}F_{\gamma\delta}F^{\gamma\delta}) \\
&= {}^{1}/_{4\pi}\,(F_{\gamma\delta}F^{\gamma\delta} - {}^{1}/_{4}\,\delta^{\alpha}{}_{\alpha}F_{\gamma\delta}F^{\gamma\delta}) \\
&= {}^{1}/_{4\pi}\,(F_{\gamma\delta}F^{\gamma\delta} - {}^{1}/_{4}\,4\,F_{\gamma\delta}F^{\gamma\delta}) \\
&= 0.
\end{aligned}
$$

Since,

$$
\begin{aligned}
\kappa T^{\alpha}{}_{\beta} &= R^{\alpha}{}_{\beta} - {}^{1}/_{2}\delta^{\alpha}{}_{\beta}R \\
\kappa T^{\alpha}{}_{\alpha} &= R^{\alpha}{}_{\alpha} - {}^{1}/_{2}\delta^{\alpha}{}_{\alpha}R \\
\kappa T &= -R \\
R &= \kappa T = 0.
\end{aligned}
$$

In 3 dimensions, however, the trace of the Kronecker delta is not 4, but 3, and we can perform the same calculations (using latin indices and substituting 2π for 4π) to get:

$$
\begin{aligned}
T^{a}{}_{a} &= {}^{1}/_{2\pi}\,(F_{cd}F^{cd} - {}^{1}/_{4}\,\delta^{a}{}_{a}F_{cd}F^{cd}) && (1.9) \\
T &= {}^{1}/_{2\pi}\,(F_{cd}F^{cd} - {}^{1}/_{4}\,3\,F_{cd}F^{cd}) \\
T &= {}^{1}/_{8\pi}\,F_{cd}F^{cd}
\end{aligned}
$$

and,

$$
\begin{aligned}
\kappa T^{a}{}_{b} &= R^{a}{}_{b} - {}^{1}/_{2}\delta^{a}{}_{b}R \\
\kappa T^{a}{}_{a} &= R^{a}{}_{a} - {}^{1}/_{2}\delta^{a}{}_{a}R \\
\kappa T &= -\,{}^{1}/_{2}\,R \\
R &= -2\kappa T = -\kappa\,{}^{1}/_{4\pi}\,F_{cd}F^{cd}. && (1.10)
\end{aligned}
$$

It is very important here that the factor of ${}^{1}/_{4}$ in equation (1.9) is independent of the number of dimensions.

It is by these equations that all gravitational and electromagnetic interactions will be governed in Flatland.

Notes

[1] J. R. Gott III and M. Alpert, G.R.G. **16**, 243 (1984).

[2] S. Deser, R. Jackiw, and G. t'Hooft, Ann. Phys. **152**, 220 (1984); S. Giddings, J. Abbot, and K. Kuchar, G.R.G. **16**, 751 (1984).

[3] M. Alpert, *General Relativity in Flatland* (Astrophysics Undergraduate Thesis: Princeton University, 1982).

Chapter 2
KALUZA-KLEIN THEORY

The original Kaluza-Klein Theory was first published in 1921 by Theodor Kaluza[1] as a way to unify Electromagnetism and Gravity (the only two forces then known), by postulating a fifth dimension with certain properties. In 1926 Oskar Klein proved that several of the constraints that Kaluza had imposed were unnecessary[2] and since then the theory has shared their names. The classic version of the theory is summed up as follows:

For curved space in 4 dimensions in the presence of an electromagnetic field, the Lagrange function is given as:

$$L = R + (\kappa/8\pi)\, F_{\mu\nu}F^{\mu\nu},$$

($\kappa/8\pi \rightarrow \kappa/2$ in systems with non-Gaussian units). From this Lagrange function can be derived both Einstein's equation and the stress-energy tensor due to electromagnetic fields by extremizing the action:

$$S = \int \sqrt{-g}\; L\, d^4x.$$

A simple unification of gravity and electromagnetism would easily be obtained by finding some curvature scalar of 5 dimensions, ${}^{(5)}R$, such that

$$\int \sqrt{-{}^{(5)}g}\;\; {}^{(5)}R\, d^5x \;=\; \int \sqrt{-g}\,(\, R + (\kappa/8\pi)\, F_{\mu\nu}F^{\mu\nu}\,)\, d^4x.$$

Kaluza discovered that this could be done by defining ${}^{(5)}g$ such that:

1) $^{(5)}g_{\mu 4} = bA_\mu$

2) $^{(5)}g_{\mu\nu} = {}^{(4)}g_{\mu\nu} + (^{(5)}g_{\mu 4})(^{(5)}g_{\nu 4})$, and

3) the entire metric is independent of x^4.

where $A^\mu = (\varphi, \mathbf{A})$ is the electromagnetic potential 4-vector. This gives:

$$L = R + {}^1/_4\,(^{(5)}g_{44})\,(b^2)\,F_{\mu\nu}F^{\mu\nu}$$

and so $^{(5)}g_{44}$ must be a positive constant, usually normalized to unity, giving:

4) $^{(5)}g_{44} = 1$,

and $b=\sqrt{\kappa/2\pi}$. The postulate that the metric is independent of x^4 makes x^4 a "cyclic" coordinate, thus conserving momentum p_4 along geodesics (see Appendix A for this proof). Kaluza postulates that p_4 is proportional to the charge of the particle,

$$p_4 \equiv \frac{q}{b} = m\,\frac{dx_4}{d\tau},$$

which lets us regard the 5-vector $\mathbf{p}$ as the "energy-momentum-charge" of the particle.

The question of why only 4 dimensions are observable to us is answered by giving the fifth dimension a very small characteristic length ℓ, smaller than any length scale measured so far. Klein suggested that this dimension be periodic in ℓ, so that any event $(t, \mathbf{x}, x^4)$ would be identified with the event $(t, \mathbf{x}, x^4+\ell)$. Here enters an interesting trick by which electric charge can be quantized. Quantum mechanically, if ℓ is small, the uncertainty principle puts limits on the momentum in that direction,

$$p_4 \simeq {}^{nh}/_{\ell}\,,$$

where h is Planck's constant, and n is any integer. This quantization of p_4 in turn quantizes charge, since $p_4 = {}^q/_b = {}^{ne}/_b$ (e = electronic charge). In fact setting the two expressions equal gives us a value for ℓ:

$$e\left(\frac{\chi}{2\pi}\right)^{-1/2} \simeq \frac{\hbar}{\ell}$$

$$\ell \simeq \frac{\hbar}{e}\left(\frac{\chi}{2\pi}\right)^{1/2} \simeq 8\times 10^{-31} \text{ cm},$$

which is very small indeed.

Aside from quantization of charge, unfortunately, the Kaluza-Klein Theory in this simple form does not provide any new insight into the workings of electromagnetism. It does, however, provide a basis for expanded theories which are still being put forward today. One variant of this theory is to remove the requirement that g_{44} be constant, letting it vary as a function of x^μ. Another variant is to let the metric be periodically dependent on x^4. More recent theories have added up to 7 or more new dimensions instead of just one, in order to account for the previously undiscovered weak and strong interactions. At this point, the most promising theories are those of 11 dimensions, the minimum needed by a Kaluza-Klein Theory to describe all four interactions, but the maximum allowed by Supersymmetric unification theories.

In this paper, instead of adding a fifth dimension to our four, we will add a fourth dimension to the three dimensional universe of Flatland. We might expect this expanded four dimensional universe to have some properties very similar to ours, but the Kaluza-Klein constraints make it sufficiently different to merit further exploration. The constraints are:

$$^{(4)}g_{33} = 1 \tag{2.1}$$

$$^{(4)}g_{m3} = bA_m \tag{2.2}$$

$$^{(4)}g_{mn} = {}^{(3)}g_{mn} + {}^{(4)}g_{m3}{}^{(4)}g_{3n} \tag{2.3}$$

$$^{(4)}g_{\mu\nu,3} = 0 \text{ for all } \mu,\nu. \tag{2.4}$$

Constraint (2.4) lets us again make the connection

$$p_3 = \text{constant} \equiv q/b. \qquad (2.5)$$

As might be expected in 3 dimensions, the constant b does not equal $\sqrt{\kappa/2\pi}$ but instead $b = \sqrt{\kappa/\pi}$. This will be proved in chapter 3. Furthermore, the newly added dimension is space-like ($g_{33} = 1$), so the four dimensions of Kaluza-Klein Flatland will share many properties of our everyday four dimensions. To emphasize this, we will call the added dimension "z" ($z \equiv x^3$).

In the next chapter we will examine one specific application of the Kaluza-Klein theory: the case of an electromagnetic wave and the motion of a test particle in this field.

Notes

[1] T. Kaluza, Sitzungsberichte, Preussische Akadamie der Wissenschaften, 966 (1921).

[2] O. Klein, Z. Phys., **37**, 896 (1926).

Chapter 3

KALUZA-KLEIN WAVE

We will apply the Kaluza-Klein theory to the case of a charged particle moving under the influence of an electromagnetic monochromatic plane wave. First we must solve Maxwell's equations in 3 dimensions in a vacuum (where $\rho=\mathbf{J}=0$).

$$\nabla\cdot E = 0 \tag{3.1}$$

$$R\nabla B - \frac{\partial E}{\partial t} = 0 \tag{3.2}$$

$$\nabla \cdot (RE) + \frac{\partial B}{\partial t} = 0 \tag{3.3}$$

We operate on equation (3.2) with the operator $\nabla\cdot R$, then operate on equation (3.3) with $\partial/\partial t$, and subtract the results to get:

$$0 = \frac{\partial^2 B}{\partial t^2} + -\nabla\cdot(RR\nabla B)$$

$$0 = \frac{\partial^2 B}{\partial t^2} + -\nabla\cdot(-\nabla B)$$

$$0 = \frac{\partial^2 B}{\partial t^2} + \nabla^2 B \tag{3.4}$$

So B satisfies the wave equation, as we would expect. The wave equation for **E** is slightly more complex to derive. First operate on equation (3.2) with $\partial/\partial t$, then on equation (3.3) with $R\nabla$ and add the two resulting equations to get:

$$0 = \frac{\partial^2 E}{\partial t^2} + R\nabla[\nabla\cdot(RE)]$$

$$= \frac{\partial^2 E}{\partial t^2} + R\nabla[\nabla\cdot(E_2\,\hat{\mathbf{x}} - E_1\hat{\mathbf{y}})]$$

$$= \frac{\partial^2 E}{\partial t^2} + R\nabla\left(\frac{\partial E_2}{\partial x} - \frac{\partial E_1}{\partial y}\right)$$

$$= \frac{\partial^2 E}{\partial t^2} + R\left[\left(\frac{\partial^2 E_2}{\partial x^2} - \frac{\partial^2 E_1}{\partial y \partial x}\right)\hat{\mathbf{x}} - \left(\frac{\partial^2 E_2}{\partial x \partial y} - \frac{\partial^2 E_1}{\partial y^2}\right)\hat{\mathbf{y}}\right]$$

but from equation (3.1),

$$0 = \frac{\partial E_1}{\partial x} + \frac{\partial E_2}{\partial y}\ ,$$

so

$$0 = \frac{\partial^2 E}{\partial t^2} + R\left[\left(\frac{\partial^2 E_2}{\partial x^2} + \frac{\partial^2 E_2}{\partial y^2}\right)\hat{\mathbf{x}} - \left(\frac{\partial^2 E_1}{\partial x^2} + \frac{\partial^2 E_1}{\partial y^2}\right)\hat{\mathbf{y}}\right]$$

$$= \frac{\partial^2 E}{\partial t^2} + \left(\frac{\partial^2 E_1}{\partial x^2} + \frac{\partial^2 E_1}{\partial y^2}\right)\hat{\mathbf{x}} + \left(\frac{\partial^2 E_2}{\partial x^2} + \frac{\partial^2 E_2}{\partial y^2}\right)\hat{\mathbf{y}}$$

or,

$$0 = \frac{\partial^2 E}{\partial t^2} + \nabla^2 E\ . \tag{3.5}$$

Again, as expected, **E** obeys the wave equation. Notice, however, that there is only one polarization possible for the electromagnetic wave: all waves traveling in the same direction can differ at most by a phase.

Let us choose a wave traveling in the $\hat{\mathbf{y}}$ direction. This results in an electromagnetic field with the following components:

$$E_1 = -E_0 \cos[\omega(t-y)]$$
$$E_2 = 0$$
$$B = E_0 \cos[\omega(t-y)]$$

For a particle of charge e (small enough not to perturb the field) and mass m (also small enough not to contribute to any gravitational attraction), this can easily be solved by standard methods to get:

$$x = (eE_0/m\omega^2)\cos\omega\tau$$
$$y = -{}^1/_8(eE_0/m)^2\,\omega^{-3}\sin 2\omega\tau \qquad (3.6)$$
$$t = \tau - {}^1/_8(eE_0/m)^2\,\omega^{-3}\sin 2\omega\tau$$

where τ = t-y. The particle moves in the shape of a figure-eight. This is the exact same solution for a particle in a plane wave in 4 dimensions traveling in the $\hat{\mathbf{y}}$ direction with $\mathbf{E} \parallel \hat{\mathbf{x}}$ (see Landau & Lifschitz[1]). We will actually solve for the motion using the Kaluza-Klein formalism.

First we would like to calculate the gravitational effects of the wave on the metric. Unfortunately this is doomed to failure for the following reason: The curvature, R_{mn}, is proportional to terms in the Energy-Momentum tensor, T_{mn}, which is in turn proportional to the Electromagnetic tensor squared, $(F_{mn})^2$, which is proportional to $E_0{}^2\cos^2[\omega(t-y)]$. That is:

$$R_{mn} \propto T_{mn} \propto (F_{mn})^2 \propto E_0{}^2\cos^2[\omega(t-y)].$$

Unfortunately, it is the second derivative of R_{mn} which is proportional to the corrections to the metric h_{mn}, so:

$$h_{mn} \propto \iint E_0{}^2\cos^2[\omega(t-y)],$$

which will always diverge when integrated over all space and time. This makes sense, because an infinite plane wave will always have an infinite amount of electromagnetic energy. Luckily, however, the contribution is 2nd order in E_0, and so can be ignored in the linear approximation. Therefore we assume the metric:

$${}^{(3)}g_{mn} \simeq \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We next need the electromagnetic potential A^m:

$$A^0 = 0$$

$$A^1 = (E_0/_{\omega})\sin[\omega(t-y)]$$

$$A^2 = 0$$

and its covariant counterpart:

$$A_0 = 0$$

$$A_1 = (E_0/\omega) \sin[\omega(t-y)]$$

$$A_2 = 0.$$

From here we can apply the Kaluza-Klein theory as formulated in chapter 2, using equations (2.1) - (2.4)

$${}^{(4)}g_{i3} = bA_i,$$

so

$${}^{(4)}g_{03} = 0$$

$${}^{(4)}g_{13} = bE_0/\omega \sin[\omega(t-y)]$$

$${}^{(4)}g_{23} = 0,$$

and

$${}^{(4)}g_{11} = {}^{(3)}g_{11} + ({}^{(4)}g_{13})^2 = 1 + (bE_0/\omega)^2 \sin^2[\omega(t-y)]$$

which gives:

$${}^{(4)}g_{\mu\nu} \simeq \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 + (bE_0/\omega)^2 \sin^2[\omega(t-y)] & 0 & (bE_0/\omega) \sin[\omega(t-y)] \\ 0 & 0 & 1 & 0 \\ 0 & (bE_0/\omega) \sin[\omega(t-y)] & 0 & 1 \end{bmatrix}.$$

It should be noted that g_{11} has one term of 2nd order, despite the fact that we just decided to only use the linearized equations. Strictly speaking, this term should be dropped, but the solutions presented above for x, y & t (equations 3.6) do contain 2nd order terms. If we want to keep any hope of calculating them we must keep some 2nd order terms. Of course if our final results are incorrect in 2nd order terms, we know why.

Next we calculate the connection coefficients:

$$\Gamma_{\alpha\beta\gamma} = {}^1/_2(g_{\alpha\beta,\gamma} + g_{\gamma\alpha,\beta} + g_{\beta\gamma,\alpha}):$$

$$\Gamma_{011} = -[\,(bE_0)^2/_\omega\,]\sin[\omega(t-y)]\cos[\omega(t-y)]$$

$$= -\Gamma_{101} = -\Gamma_{110} = -\Gamma_{211} = \Gamma_{121} = \Gamma_{112}$$

$$\Gamma_{013} = -{}^1/_2 bE_0 \cos[\omega(t-y)]$$

$$= \Gamma_{031} = -\Gamma_{103} = -\Gamma_{301} = -\Gamma_{130} = -\Gamma_{310}$$

$$= -\Gamma_{213} = -\Gamma_{231} = \Gamma_{123} = \Gamma_{321} = \Gamma_{132} = \Gamma_{312}\,.$$

All other $\Gamma_{\alpha\beta\gamma} = 0$. In order to find $\Gamma^\alpha{}_{\beta\gamma}$ we need $g^{\mu\nu}$, which is defined such that $g^{\mu\nu}g_{\nu\lambda} = \delta^\mu{}_\lambda$:

$$g^{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -({}^{bE_0}/_\omega)\sin[\omega(t-y)] \\ 0 & 0 & 1 & 0 \\ 0 & -({}^{bE_0}/_\omega)\sin[\omega(t-y)] & 0 & 1+({}^{bE_0}/_\omega)^2\sin^2[\omega(t-y)] \end{bmatrix}$$

Now we can calculate $\Gamma^\alpha{}_{\beta\gamma} = g^{\alpha\mu}\Gamma_{\mu\beta\gamma}$, keeping in mind that $\Gamma^\alpha{}_{\beta\gamma} = \Gamma^\alpha{}_{\gamma\beta}$.

$$\Gamma^0{}_{11} = g^{00}\Gamma_{011} = [\,(bE_0)^2/_\omega\,]\sin[\omega(t-y)]\cos[\omega(t-y)]$$

$$\Gamma^0{}_{13} = g^{00}\Gamma_{013} = {}^1/_2\, bE_0\cos[\omega(t-y)]$$

$$= \Gamma^0{}_{31} \text{ by symmetry}$$

$$\Gamma^1{}_{01} = g^{11}\Gamma_{101} + g^{13}\Gamma_{301}$$

$$= [\,(bE_0)^2/_\omega\,]\sin[\omega(t-y)]\cos[\omega(t-y)]$$

$$+\{-({}^{bE_0}/_\omega)\sin[\omega(t-y)]\}\times\{{}^1/_2 bE_0\cos[\omega(t-y)]\}$$

$$= {}^1/_2[\,(bE_0)^2/_\omega\,]\sin[\omega(t-y)]\cos[\omega(t-y)]$$

$$= \Gamma^1{}_{10} \text{ by symmetry}$$

$$\Gamma^1{}_{03} = g^{11}\Gamma_{103} = {}^1/_2 bE_0\cos[\omega(t-y)]$$

$$= \Gamma^1{}_{30} \text{ by symmetry}$$

$$\Gamma^1{}_{12} = g^{11}\Gamma_{112} + g^{13}\Gamma_{312}$$

$$= -[\,(bE_0)^2/_\omega\,]\sin[\omega(t-y)]\cos[\omega(t-y)]$$

$$+\{-({}^{bE_0}/_\omega)\sin[\omega(t-y)]\}\times\{-{}^1/_2 bE_0\cos[\omega(t-y)]\}$$

$= -{}^{1}/_{2}[\, (bE_0)^2/_{\omega}\,] \sin[\omega(t-y)] \cos[\omega(t-y)]$

$= \Gamma^1{}_{12}$ by symmetry

$\Gamma^1{}_{23} = g^{11}\Gamma_{123} = -{}^{1}/_{2} bE_0 \cos[\omega(t-y)]$

$= \Gamma^1{}_{32}$ by symmetry

$\Gamma^2{}_{11} = g^{22}\Gamma_{211} = [\, (bE_0)^2/_{\omega}\,] \sin[\omega(t-y)] \cos[\omega(t-y)]$

$\Gamma^2{}_{13} = g^{22}\Gamma_{213} = {}^{1}/_{2} bE_0 \cos[\omega(t-y)]$

$= \Gamma^2{}_{31}$ by symmetry

$\Gamma^3{}_{01} = g^{33}\Gamma_{301} + g^{31}\Gamma_{101}$

$=\{1 + ({}^{bE_0}/_{\omega})^2 \sin^2[\omega(t-y)]\} \times \{{}^{1}/_{2} bE_0 \cos[\omega(t-y)]\}$

$- \{({}^{bE_0}/_{\omega}) \sin[\omega(t-y)]\} \times \{[(bE_0)^2/_{\omega}] \sin[\omega(t-y)] \cos[\omega(t-y)]\}$

$= \{{}^{1}/_{2} bE_0 \cos[\omega(t-y)]\} \times \{\, 1 - ({}^{bE_0}/_{\omega})^2 \sin^2[\omega(t-y)]\, \}$

$\simeq {}^{1}/_{2} bE_0 \cos[\omega(t-y)]$

$= \Gamma^3{}_{10}$ by symmetry

$\Gamma^3{}_{12} = g^{33}\Gamma_{312} + g^{31}\Gamma_{112}$

$=\{\, 1 + ({}^{bE_0}/_{\omega})^2 \sin^2[\omega(t-y)]\, \} \times \{-{}^{1}/_{2} bE_0 \cos[\omega(t-y)]\}$

$+ \{\, ({}^{bE_0}/_{\omega}) \sin[\omega(t-y)]\, \}$

$\times \{\, -[\, (bE_0)^2/_{\omega}\,] \sin[\omega(t-y)] \cos[\omega(t-y)]\, \}$

$= -\{{}^{1}/_{2} bE_0 \cos[\omega(t-y)]\} \times \{\, 1 - ({}^{bE_0}/_{\omega})^2 \sin^2[\omega(t-y)]\, \}$

$\simeq -{}^{1}/_{2} bE_0 \cos[\omega(t-y)]$

$= \Gamma^3{}_{21}$ by symmetry,

and all other $\Gamma^{\alpha}{}_{\beta\gamma} = 0$. Now we are ready to use the geodesic equation. There are no external forces since in the Kaluza-Klein formalism the effects of electromagnetism are mimicked by the extra dimension.

$$0 \quad = \frac{d^2x^{\mu}}{d\tau^2} + \Gamma^{\mu}{}_{\alpha\beta} \frac{dx^{\alpha}}{d\tau} \frac{dx^{\beta}}{d\tau}$$

For convenience of notation, let $\dot{f} = \frac{df}{d\tau}$ (not $\frac{df}{dt}$).

$$0 \quad = \frac{d^2x^0}{d\tau^2} + \Gamma^0{}_{11} \frac{dx^1}{d\tau}\frac{dx^1}{d\tau} + 2\Gamma^0{}_{13} \frac{dx^1}{d\tau}\frac{dx^3}{d\tau}$$

$$= \ddot{t} + [(bE_0)^2/\omega]\sin[\omega(t-y)]\cos[\omega(t-y)]\,\dot{x}^2 + 2\ {}^1/_2 bE_0 \cos[\omega(t-y)]\dot{x}\dot{z}$$

$$= \ddot{t} + bE_0\,\dot{x}\cos[\omega(t-y)] \times \{ (bE_0/\omega)\,\dot{x}\sin[\omega(t-y)] + \dot{z} \}$$

$$0 = \frac{d^2x^1}{d\tau^2} + 2\Gamma^1{}_{01}\frac{dx^0}{d\tau}\frac{dx^1}{d\tau} + 2\Gamma^1{}_{03}\frac{dx^0}{d\tau}\frac{dx^3}{d\tau} + 2\Gamma^1{}_{12}\frac{dx^1}{d\tau}\frac{dx^2}{d\tau} + 2\Gamma^1{}_{23}\frac{dx^2}{d\tau}\frac{dx^3}{d\tau}$$

$$= \ddot{x} + 2\cdot{}^1/_2\,[\,(bE_0)^2/_\omega\,]\sin[\omega(t-y)]\cos[\omega(t-y)]\,\dot{t}\dot{x}$$

$$+ 2\cdot{}^1/_2\,bE_0\cos[\omega(t-y)]\,\dot{t}\dot{z}$$

$$- 2\cdot{}^1/_2\,[\,(bE_0)^2/_\omega\,]\sin[\omega(t-y)]\cos[\omega(t-y)]\,\dot{x}\dot{y}$$

$$- 2\cdot{}^1/_2\,bE_0\cos[\omega(t-y)]\ \dot{y}\dot{z}$$

$$= \ddot{x} + bE_0\,(\dot{t}-\dot{y})\cos[\omega(t-y)] \times \{ (bE_0/_\omega)\,\dot{x}\sin[\omega(t-y)] + \dot{z}\}$$

$$0 = \frac{d^2x^2}{d\tau^2} + \Gamma^2{}_{11}\frac{dx^1}{d\tau}\frac{dx^1}{d\tau} + 2\Gamma^2{}_{13}\frac{dx^1}{d\tau}\frac{dx^3}{d\tau}$$

$$= \ddot{y} + bE_0\,\dot{x}\cos[\omega(t-y)]\,\{ (bE_0/_\omega)\dot{x}\sin[\omega(t-y)] + \dot{z} \}$$

$$0 = \frac{d^2x^3}{d\tau^2} + 2\Gamma^3{}_{01}\frac{dx^0}{d\tau}\frac{dx^1}{d\tau} + 2\Gamma^3{}_{12}\frac{dx^1}{d\tau}\frac{dx^2}{d\tau}$$

$$= \ddot{z} + 2\cdot{}^1/_2 bE_0\cos[\omega(t-y)]\,\dot{t}\dot{x} - 2\cdot{}^1/_2 bE_0\cos[\omega(t-y)]\,\dot{x}\dot{y}$$

$$= \ddot{z} + bE_0\,\dot{x}\cos[\omega(t-y)]$$

$$\ddot{t}-\ddot{y} = 0 \quad \Rightarrow \dot{t}-\dot{y} = \text{constant} \equiv 1 \quad \Rightarrow t-y = \tau.$$

Now since $x^3=z$ is a cyclic coordinate ($g_{\mu\nu,3}= 0$), we know that the z component of the covariant 4-velocity is conserved, and we make this component proportional to the charge, as in equation (2.5).

$$\frac{dx_3}{d\tau} = \text{constant} \equiv \frac{e}{bm} = g_{3\mu}\frac{dx^\mu}{d\tau} = g_{31}\frac{dx}{d\tau} + g_{33}\frac{dz}{d\tau}$$

$$\frac{e}{bm} = \left(\frac{bE_0}{\omega}\right)\dot{x}\sin[\omega(t-y)] + \dot{z} = \left(\frac{bE_0}{\omega}\right)\dot{x}\sin\omega\tau + \dot{z}$$

$$\ddot{x} = -(eE_0/_m)\cos\omega\tau$$

$$\dot{x} = -(eE_0/_{m\omega})\sin\omega\tau$$

$$x = (eE_0/_{m\omega^2})\cos\omega\tau$$

$$\ddot{t} = [\,(eE_0/m)^2/_\omega\,]\sin\omega\tau\cos\omega\tau$$

$$\dot{t} = -(eE_0/2m\omega)^2 \cos 2\omega\tau + c_1$$
$$t = -\tfrac{1}{8}(eE_0/m)^2\, \omega^{-3} \sin 2\omega\tau + c_1\tau + c_2$$

since $t \to \tau$ as $e \to 0$

$$t = \tau - \tfrac{1}{8}(eE_0/m)^2\, \omega^{-3} \sin 2\omega\tau$$
$$y = t - \tau = -\tfrac{1}{8}(eE_0/m)^2\, \omega^{-3} \sin 2\omega\tau$$

which gives us the results desired in equation (3.6), even to 2^{nd} order. Unfortunately, when we calculate z from conservation of momentum:

$$e/bm = (bE_0/\omega)\, \dot{x} \sin\omega\tau + \dot{z}$$

we get

$$\dot{z} = e/bm + (bE_0/\omega)\,(eE_0/m\omega) \sin\omega\tau \sin\omega\tau$$
$$\ddot{z} = 2(\, ebE_0^2/m\omega\,) \sin\omega\tau \cos\omega\tau, \qquad (3.7)$$

but from the geodesic equation:

$$\ddot{z} = bE_0\, (-eE_0/m\omega) \sin\omega\tau \cos\omega\tau$$
$$= -\,(\, ebE_0^2/m\omega\,) \sin\omega\tau \cos\omega\tau,$$

which is inconsistent with (3.7). So here the equations are not valid to 2^{nd} order in z, and all we know is that

$$z = e\tau/bm$$

to first order only.

Now compare this Kaluza-Klein metric to the case of a gravitational wave in the standard 4 dimensional case. Use the linearized Kaluza-Klein metric:

$$^{(4)}g_{\mu\nu} \simeq \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & (bE_0/\omega)\sin[\omega(t-y)] \\ 0 & 0 & 1 & 0 \\ 0 & (bE_0/\omega)\sin[\omega(t-y)] & 0 & 1 \end{bmatrix}.$$

or $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu}$ is given by:

$$h_{\mu\nu} \simeq \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (bE_0/_\omega)\sin[\omega(t-y)] \\ 0 & 0 & 0 & 0 \\ 0 & (bE_0/_\omega)\sin[\omega(t-y)] & 0 & 0 \end{bmatrix}.$$

This is the form of a linearized gravitational wave with "×" polarization, and it is exactly what we should have predicted, because in the 3 dimensional case, there are no gravitational waves, only electromagnetic waves. In the Kaluza-Klein 4 dimensional case, however, there are no electromagnetic waves, so anything traveling at the speed of light with the form of a wave must be a gravitational wave. Usually in 4 dimensions there are two polarizations of a gravitational wave: the "×" polarization & the "+" polarization. In 3 dimensions there is only one polarization for an electromagnetic wave. What happens to the "+" polarization? For "+" polarization, $h_{\mu\nu}$ must be of the form:

$$h_{\mu\nu} \simeq \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & K\sin[\omega(t-y)] & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & K\sin[\omega(t-y)] \end{bmatrix}.$$

This would mean:

$$g_{33} = \eta_{33} + h_{33} = 1 + K\sin[\omega(t-y)] ,$$

but the Kaluza-Klein formalism states that g_{33} must be a constant, or else the strength of the electromagnetic force would vary with distance. We conclude that there is only one polarization allowed in the Kaluza-Klein 4 dimensional case, the "×" polarization, and that it corresponds to an electromagnetic wave in 3 dimensions.

Now we calculate the value of the constant b, $(=\sqrt{\kappa/2\pi}$ in the higher dimensional case). Let us calculate the other relevant quantities of the system:

$$g^{\mu\nu} \simeq \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -(bE_0/\omega)\sin[\omega(t-y)] \\ 0 & 0 & 1 & 0 \\ 0 & -(bE_0/\omega)\sin[\omega(t-y)] & 0 & 1 \end{bmatrix}$$

$$\Gamma^0{}_{13} = \Gamma^1{}_{03} = -\Gamma^1{}_{23} = \Gamma^2{}_{13} = -\Gamma^3{}_{12} = \Gamma^3{}_{01} = {}^1/_2 bE_0\cos\omega\tau$$

$$=\Gamma^0{}_{31} = \Gamma^1{}_{30} = -\Gamma^1{}_{32} = \Gamma^2{}_{31} = -\Gamma^3{}_{21} = \Gamma^3{}_{10}$$

(all other $\Gamma^\alpha{}_{\beta\gamma} = 0$)

$$R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu} = \Gamma^\alpha{}_{\mu\nu,\alpha} - \Gamma^\alpha{}_{\mu\alpha,\nu} + \Gamma^\alpha{}_{\beta\alpha}\Gamma^\beta{}_{\mu\nu} - \Gamma^\alpha{}_{\beta\nu}\Gamma^\beta{}_{\alpha\mu}$$

$$= \Gamma^\alpha{}_{\mu\nu,\alpha} - \Gamma^\alpha{}_{\beta\nu}\Gamma^\beta{}_{\alpha\mu}$$

$$R_{00} = -\Gamma^\alpha{}_{\beta 0}\Gamma^\beta{}_{\alpha 0} = -2\Gamma^1{}_{30}\Gamma^3{}_{10}$$

$$= -{}^1/_2(bE_0\cos\omega\tau)^2$$

$$R_{01} = 0$$

$$R_{02} = -\Gamma^\alpha{}_{\beta 2}\Gamma^\beta{}_{\alpha 0} = -\Gamma^1{}_{32}\Gamma^3{}_{10} - \Gamma^3{}_{12}\Gamma^1{}_{30}$$

$$= {}^1/_4(bE_0\cos\omega\tau)^2 + {}^1/_4(bE_0\cos\omega\tau)^2$$

$$= {}^1/_2(bE_0\cos\omega\tau)^2$$

$$R_{03} = 0$$

$$R_{11} = -\Gamma^\alpha{}_{\beta 1}\Gamma^\beta{}_{\alpha 1} = -2\Gamma^0{}_{31}\Gamma^3{}_{01} - 2\Gamma^2{}_{31}\Gamma^3{}_{21}$$

$$= -{}^1/_2(bE_0\cos\omega\tau)^2 + {}^1/_2(bE_0\cos\omega\tau)^2$$

$$= 0$$

$$R_{12} = 0$$

$$R_{13} = 0$$

$$R_{22} = -\Gamma^\alpha{}_{\beta 2}\Gamma^\beta{}_{\alpha 2} = -2\Gamma^1{}_{32}\Gamma^3{}_{12} = -{}^1/_2(bE_0\cos\omega\tau)^2$$

$$R_{23} = 0$$

$$R_{33} = -\Gamma^{\alpha}{}_{\beta 3}\Gamma^{\beta}{}_{\alpha 3} = -2\Gamma^{0}{}_{13}\Gamma^{1}{}_{03} - 2\Gamma^{0}{}_{13}\Gamma^{1}{}_{03}$$

$$= -{}^{1}/_{2}(bE_0\cos\omega\tau)^2 + {}^{1}/_{2}(bE_0\cos\omega\tau)^2$$

$$= 0$$

and $R_{\mu\nu} = R_{\nu\mu}$. This can be expressed easily in matrix form:

$$R_{\mu\nu} = {}^{1}/_{2}(bE_0\cos\omega\tau)^2 \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now we solve for $R^{\mu}{}_{\nu} = g^{\mu\alpha}R_{\alpha\nu}$:

$$R^{0}{}_{0} = g^{00}\, R_{00} = +{}^{1}/_{2}(bE_0\cos\omega\tau)^2$$

$$R^{0}{}_{2} = g^{00}\, R_{02} = -{}^{1}/_{2}(bE_0\cos\omega\tau)^2$$

$$R^{2}{}_{0} = g^{22}\, R_{20} = +{}^{1}/_{2}(bE_0\cos\omega\tau)^2$$

$$R^{2}{}_{2} = g^{22}\, R_{22} = -{}^{1}/_{2}(bE_0\cos\omega\tau)^2$$

or:

$$R^{\mu}{}_{\nu} = {}^{1}/_{2}(bE_0\cos\omega\tau)^2 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The Ricci curvature scalar vanishes identically:

$$R = R^{\alpha}{}_{\alpha} = R^{0}{}_{0} + R^{2}{}_{2} = 0,$$

so:

$$\kappa T^{\mu}{}_{\nu} = G^{\mu}{}_{\nu} = R^{\mu}{}_{\nu} + {}^{1}/_{2}\delta^{\mu}{}_{\nu}R = R^{\mu}{}_{\nu} = {}^{1}/_{2}(bE_0\cos\omega\tau)^2 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (3.8)$$

Now let us calculate $T^{m}{}_{n}$, using equation (1.8):

$$T^{m}{}_{n} = {}^{1}/_{2\pi}\,(g_{ab}F^{ma}F_{n}{}^{b} - {}^{1}/_{4}\,\delta^{m}{}_{n}F_{ab}F^{ab})$$

$$= {}^{1}/_{2\pi}\,(F^{am}F_{an} - {}^{1}/_{4}\,\delta^{m}{}_{n}F_{ab}F^{ab})$$

Let us first solve for the second part, $F_{ab}F^{ab}$, recalling that

$$F^{01} = -F^{10} = E_1 = -E_0\cos\omega\tau$$

$$= -F_{01} = F_{10}$$

$$F^{12} = -F^{21} = B = E_0 \cos\omega\tau$$

$$= F_{12} = -F_{21},$$

so,

$$F_{ab}F^{ab} = F_{01}F^{01} + F_{10}F^{10} + F_{12}F^{12} + F_{21}F^{21} = 0,$$

and,

$$T^m{}_n = {}^1/_{2\pi}\, F^{am}F_{an}:$$

$$T^0{}_0 = {}^1/_{2\pi}\,(F^{01}F_{01}) = +{}^1/_{2\pi}\,(E_0\cos\omega\tau)^2$$

$$T^0{}_2 = {}^1/_{2\pi}\,(F^{01}F_{21}) = +{}^1/_{2\pi}\,(E_0\cos\omega\tau)^2$$

$$T^2{}_0 = {}^1/_{2\pi}\,(F^{21}F_{01}) = -{}^1/_{2\pi}\,(E_0\cos\omega\tau)^2$$

$$T^2{}_2 = {}^1/_{2\pi}\,(F^{21}F_{21}) = -{}^1/_{2\pi}\,(E_0\cos\omega\tau)^2,$$

and all the rest of $T^m{}_n$ are identically zero. This gives:

$$T^m{}_n = {}^1/_{2\pi}\,(E_0\cos\omega\tau)^2 \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}.$$

Comparing this to equation (3.8), it seems most natural to identify

$$b^2 = \pi/\chi$$

$$b = \sqrt{\pi/\chi},$$

and so we have determined the constant b.

Notes

[1] L.D. Landau and E.M. Lifshitz, *The Classical Theory of Fields*, (Addison Wesley: Reading, MA 1962).

Chapter 4

STATIONARY CHARGE

The case of a single stationary charge in Flatland is discussed extensively by Alpert,[1] so I will only briefly present his conclusions. A massless particle of charge Q at r=0 causes a radial field of $\mathbf{E} = {}^{Q}/_{r}\ \hat{\mathbf{r}}$ (analogous to the $\mathbf{E} = {}^{Q}/_{r^2}\ \hat{\mathbf{r}}$ field in 3 spatial dimensions). This ${}^{1}/_{r}$ dependence holds exactly even as space-time curves in the presence of the electric field energy, as long as the coordinate r is defined in terms of a circumferential radius. The metric in this space is given by:

$$ds^2 = \frac{\kappa Q^2}{2\pi} \ln\left(\frac{r}{r_c}\right) dt^2 - \frac{2\pi}{\kappa Q^2}\left[\ \ln\left(\frac{r}{r_c}\right)\right]^{-1} dr^2 + r^2 d\varphi.$$

(This metric was arrived at by assuming only that the metric was of the type $ds^2 = A(r)dt^2 + B(r)dr^2 + r^2d\varphi^2$ and that the electric field was radial.) The parameter r_c, which determines the scale of the system, is analogous to R=2GM in the Schwarzchild metric. It is interesting to note, however, that r_c is independent of Q, a dimensionless quantity (actually it is κQ^2 that is dimensionless, but we are taking κ as dimensionless as well). It is easily checked that t is timelike and r is spacelike only for $r<r_c$, but discussion of this will be delayed until chapter 5.

First let us analyze the metric. Define the radius r_m:

$$r_m = r_c \exp(-2\pi/_{KQ^2}) \qquad (4.1)$$

Note that $r_m < r_c$ always. This is the radius at which $g_{00}=-1$ and $g_{11}=1$, or

$$ds^2 = -dt^2 + dr^2 + r^2 d\varphi^2.$$

Let us look at the behavior of the metric at this and other important points:

as $r \to 0$:	$g_{tt} \to -\infty$	$g_{rr} \to 0+$
at $r=r_m$:	$g_{tt} = -1$	$g_{rr} = 1$
as $r \to r_c-$:	$g_{tt} \to 0-$	$g_{rr} \to \infty$
as $r \to r_c+$:	$g_{tt} \to 0+$	$g_{rr} \to -\infty$
as $r \to \infty$:	$g_{tt} \to \infty$	$g_{rr} \to 0-$

One of the clearest ways to view the properties of a metric is its imbedding diagram. For $r<r_m$ the curvature of the metric is negative, and we must imbed the metric in Minkowski space instead of Euclidean space, so the horizontal axis is imaginary.

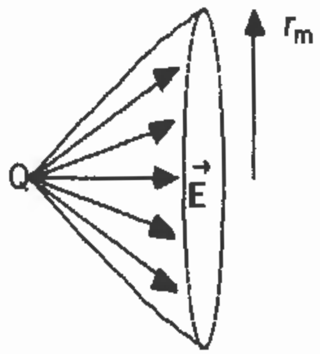


Diagram 4.1

(It should be noted here that Alpert did not allow any electric field inside the radius r_m since this produces "negative mass" by giving angle excess instead of angle deficit. Since there is no reason here to avoid angle excess, we will not worry about this restriction.) For $r_m<r<r_c$ the metric can be imbedded in ordinary Euclidean space. At $r=r_c$, however, Alpert

calculated that the mass and angle deficit would become imaginary, which must be avoided. Alpert therefore claimed that the universe must stop growing at $r=r_c$ and must then turn around and shrink in r back to r_m (and beyond to r=0), where there would be another charge with opposite sign, -Q. This results in imbedding diagram 4.2.

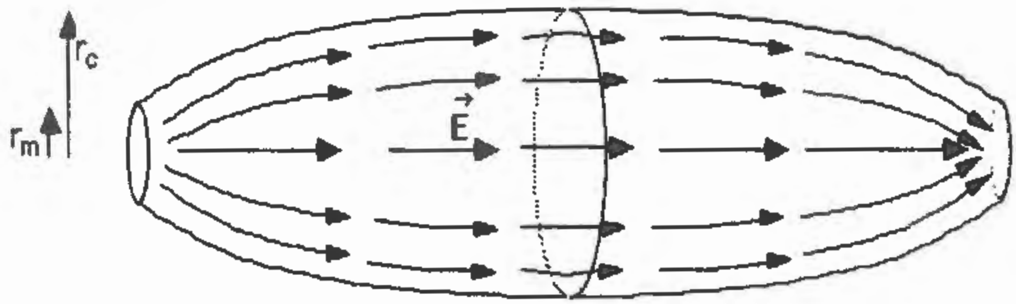


Diagram 4.2

Since the function that represents the curve in both diagrams is continuous at r_m, we can join the two into one diagram, representing all of Alpert's universe. The important features of this universe are that it is static, closed, and electrically neutral.

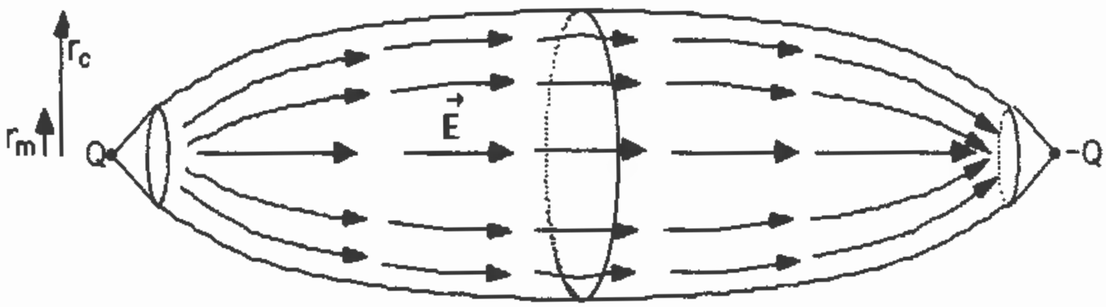


Diagram 4.3

As will be shown in chapter 5, this view of the universe is partially correct, but incomplete.

We can solve for the motion of a charged, massive particle in the gravitational and electromagnetic fields of this universe using the geodesic equations of General Relativity (again, assume small mass m and charge e so that the metric is not sufficiently perturbed). Recalling that the metric is given by

$$ds^2 = \frac{\kappa Q^2}{2\pi}\ln\left(\frac{r}{r_c}\right)dt^2 - \frac{2\pi}{\kappa Q^2}\left[\ln\left(\frac{r}{r_c}\right)\right]^{-1}dr^2 + r^2 d\varphi,$$

this gives us

$$g_{00} = \frac{\kappa Q^2}{2\pi}\ln\left(\frac{r}{r_c}\right) = (g^{00})^{-1}$$

$$g_{11} = -\frac{2\pi}{\kappa Q^2}\left[\ln\left(\frac{r}{r_c}\right)\right]^{-1} = (g^{11})^{-1}$$

$$g_{22} = r^2 = (g^{22})^{-1}$$

and all other $g_{mn}=0$. Next we calculate the components of Γ_{abc}:

$$\Gamma_{001} = \frac{\kappa Q^2}{2\pi}\frac{1}{2r}$$

$$= \Gamma_{010} = -\Gamma_{100}$$

$$\Gamma_{111} = \frac{2\pi}{\kappa Q^2}\frac{1}{2r}\left[\ln\left(\frac{r}{r_c}\right)\right]^{-2}$$

$$\Gamma_{221} = r$$

$$= \Gamma_{212} = -\Gamma_{122},$$

and all other $\Gamma_{abc} = 0$. The calculation of $\Gamma^a{}_{bc}$ is especially easy when g_{mn} is diagonal:

$$\Gamma^0{}_{01} = g^{00}\Gamma_{001} = \frac{2\pi}{\kappa Q^2}\left[\ln\left(\frac{r}{r_c}\right)\right]^{-1}\frac{\kappa Q^2}{2\pi}\frac{1}{2r} = \frac{1}{2r}\left[\ln\left(\frac{r}{r_c}\right)\right]^{-1}$$

$$= \Gamma^0{}_{10}$$

$$\Gamma^1{}_{00} = g^{11}\Gamma_{100} = -\frac{\kappa Q^2}{2\pi}\ln\left(\frac{r}{r_c}\right)\frac{-\kappa Q^2}{2\pi}\frac{1}{2r} = \left(\frac{\kappa Q^2}{2\pi}\right)^2\frac{1}{2r}\ln\left(\frac{r}{r_c}\right)$$

$$\Gamma^1{}_{11} = g^{11}\Gamma_{111} = -\frac{\kappa Q^2}{2\pi}\ln\left(\frac{r}{r_c}\right)\frac{2\pi}{\kappa Q^2}\frac{1}{2r}\left[\ln\left(\frac{r}{r_c}\right)\right]^{-2} = \frac{-1}{2r}\left[\ln\left(\frac{r}{r_c}\right)\right]^{-1}$$

$$\Gamma^1{}_{22} = g^{11}\Gamma_{122} = -\frac{\kappa Q^2}{2\pi}\ln\left(\frac{r}{r_c}\right)(-r) = \frac{\kappa Q^2}{2\pi}r\ln\left(\frac{r}{r_c}\right)$$

$$\Gamma^2{}_{12} = g^{22}\Gamma_{212} = ({}^1/_{r^2})(r) = {}^1/_r$$

$$= \Gamma^2{}_{21},$$

and all other $\Gamma^a{}_{bc} = 0$.

In an electromagnetic force field the equation of geodesic motion is:

$$\frac{e}{m} F^m{}_a \frac{dx^a}{d\tau} = \frac{d^2x^m}{d\tau^2} + \Gamma^m{}_{ab} \frac{dx^a}{d\tau}\frac{dx^b}{d\tau}$$

$$\frac{e}{m} g_{ac} F^{mc} \frac{dx^a}{d\tau} = \frac{d^2x^m}{d\tau^2} + \Gamma^m{}_{ab} \frac{dx^a}{d\tau}\frac{dx^b}{d\tau}.$$

For a=2 ($x^2=\varphi$):

$$0 \quad = \frac{d^2\varphi}{d\tau^2} + \Gamma^2_{ab} \frac{dx^a}{d\tau}\frac{dx^b}{d\tau}.$$

There is one simplification we can make here. Since there is no force in the φ direction, and since the metric is independent of φ, the φ component of the covariant momentum is conserved (see Appendix A).

$$\frac{dx_2}{d\tau} = \text{constant} \equiv u_2 = g_{22}\frac{dx^2}{d\tau} = r^2 \frac{d\varphi}{d\tau}$$

$$\frac{d\varphi}{d\tau} = u_2 \cdot {}^1/_{r^2}$$

We do not have this convenience in the other directions.

$$\frac{e}{m} g_{11} F^{01} \frac{dx^1}{d\tau} = \frac{d^2x^0}{d\tau^2} + 2\Gamma^0{}_{01}\frac{dx^0}{d\tau}\frac{dx^1}{d\tau}$$

$$\frac{e}{m}\frac{2\pi}{\kappa Q^2} \ln^{-1}\left(\frac{r}{r_c}\right)\left(\frac{-Q}{r}\right)\frac{dr}{d\tau} = \frac{d^2t}{d\tau^2} + \frac{1}{r}\left[\ln\left(\frac{r}{r_c}\right)\right]^{-1}\frac{dt}{d\tau}\frac{dr}{d\tau}$$

$$0 = \frac{d^2t}{d\tau^2} + \frac{e}{m}\frac{2\pi}{\kappa Q^2} \ln^{-1}\left(\frac{r}{r_c}\right)\left(\frac{Q}{r}\right)\frac{dr}{d\tau} + \frac{1}{r}\left[\ln\left(\frac{r}{r_c}\right)\right]^{-1}\frac{dt}{d\tau}\frac{dr}{d\tau}$$

and,

$$\frac{e}{m} g_{00} F^{10}\frac{dx^0}{d\tau} = \frac{d^2x^1}{d\tau^2} + \Gamma^1{}_{00}\frac{dx^0}{d\tau}\frac{dx^0}{d\tau} + \Gamma^1{}_{11}\frac{dx^1}{d\tau}\frac{dx^1}{d\tau} + \Gamma^1{}_{22}\frac{dx^2}{d\tau}\frac{dx^2}{d\tau}$$

$$0 = \frac{d^2r}{d\tau^2} + \frac{e}{m}\frac{\kappa Q^2}{2\pi}\ln\left(\frac{r}{r_c}\right)\frac{Q}{r}\frac{dt}{d\tau} + \left(\frac{\kappa Q^2}{2\pi}\right)^2\frac{1}{2r}\ln\left(\frac{r}{r_c}\right)\frac{dt}{d\tau}^2$$

$$- \frac{1}{2r}\left[\ln\left(\frac{r}{r_c}\right)\right]^{-1}\frac{dr}{d\tau}^2 + \frac{\kappa Q^2}{2\pi} r \ln\left(\frac{r}{r_c}\right)\left(u_2 \frac{1}{r^2}\right)^2 \qquad (4.2)$$

This gives us two equations in two unknowns (r & t), but both equations are extremely nonlinear in r and not easy to solve. We shall forgo solving the equations in general because of their difficulty. There is, however, one particular case which is both interesting and readily soluble: the case of a stationary charged particle for which the gravitational "force" is exactly balanced by the electric force. In this case, we use equation (4.2) with r=constant, to get:

$$0 \quad = \frac{e}{m}\frac{\kappa Q^2}{2\pi}\ln\left(\frac{r}{r_c}\right)\frac{Q}{r}\frac{dt}{d\tau} + \left(\frac{\kappa Q^2}{2\pi}\right)^2\frac{1}{2r}\ln\left(\frac{r}{r_c}\right)\frac{dt}{d\tau}^2$$

We can determine $^{dt}/_{d\tau}$ from the metric, using the fact that $d\tau^2=-ds^2$ in the case that $dr=d\varphi=0$. This gives us:

$$\frac{dt}{d\tau} = \left(\frac{2\pi}{\kappa Q^2}\right)^{1/2}\left[-\ln\left(\frac{r}{r_c}\right)\right]^{-1/2}$$

so:

$$0 \quad = \frac{e}{m}\left(\frac{\kappa Q^2}{2\pi}\right)^{1/2}\left[-\ln\left(\frac{r}{r_c}\right)\right]^{1/2}\frac{Q}{r} + \frac{\kappa Q^2}{2\pi}\frac{1}{2r}$$

$$\frac{m}{e} = -2\frac{\sqrt{2\pi}}{\sqrt{\kappa}}\left[-\ln\left(\frac{r}{r_c}\right)\right]^{1/2} \quad = V(r)$$

where V(r) is the electrostatic potential measured from r_c, which will be proved later in this chapter (equation 4.3). This reduces to the extremely simple equation eV=-m for a stationary test particle. First of all note the sign: the charge must be negative (attracted to the charge Q) in order to remain stationary. This mean that the test particle would fly away from r=0 if it were not charged. The gravitational "well" outside the charge is upside-down. This should not be completely unexpected, as we are fully aware that there is no Newtonian limit in 3 dimensions, and so we should not expect Newtonian-like results. We must also remember there is mass/energy from the electric field produced by the oppositely charged source, $-Q$, at "the other" $r=0$, which also contributes to gravitational interactions.

As promised above, we now calculate the potential, which is defined with reference to some from arbitrary radius $r_a < r_c$:

$$V(r') = \int_{r'}^{r_a} \mathbf{E}\cdot\mathbf{ds} = \int_{r'}^{r_a} Er\,\sqrt{g_{11}}\;dr \quad (\text{for } dt=d\varphi=0)$$

$$V(r') = \int_{r'}^{r_a} \frac{Q}{r}\left(\frac{2\pi}{\kappa Q^2}\right)^{1/2} \frac{dr}{[-\ln(r/r_c)]^{1/2}}$$

$$V(r) = 2\frac{\sqrt{2\pi}}{\sqrt{\kappa}}\left(\left[-\ln\left(\frac{r}{r_c}\right)\right]^{1/2} - \left[-\ln\left(\frac{r_a}{r_c}\right)\right]^{1/2}\right)$$

Note that this is independent of Q(!). It seems clear that an obvious point of reference from which to determine potential would be to set $r_a = r_c$, so

$$V(r) = 2\frac{\sqrt{2\pi}}{\sqrt{\kappa}}\left[-\ln\left(\frac{r}{r_c}\right)\right]^{1/2} \qquad (4.3)$$

as stated above. Any explorer starting a quest to solve this case using the Kaluza-Klein formalism should use this quantity to determine g_{03}, and proceed from there.

For completeness, let us continue solving for important quantities of this system. Recalling the definition of $R^a{}_{bcd}$ and R_{mn}:

$$R^a{}_{bcd} = \Gamma^a{}_{bd,c} - \Gamma^a{}_{bc,d} + \Gamma^a{}_{mc}\Gamma^m{}_{bd} - \Gamma^a{}_{md}\Gamma^m{}_{bc}$$

$$R_{mn} = R^a{}_{man} = \Gamma^a{}_{mn,a} - \Gamma^a{}_{ma,n} + \Gamma^a{}_{ba}\Gamma^b{}_{mn} - \Gamma^a{}_{mb}\Gamma^b{}_{na}$$

$$= \Gamma^1{}_{mn,1} - \delta_{1n}\Gamma^a{}_{ma,n} + \Gamma^a{}_{ba}\Gamma^b{}_{mn} - \Gamma^a{}_{mb}\Gamma^b{}_{na}$$

since all g_{mn} are functions of r ($=x^1$) only. Now we solve for the individual components:

$$R_{00} = \Gamma^1{}_{00,1} + \Gamma^1{}_{00}(\Gamma^0{}_{10} + \Gamma^1{}_{11} + \Gamma^2{}_{12}) - 2\Gamma^0{}_{01}\Gamma^1{}_{00}$$

$$= \Gamma^1{}_{00,1} + \Gamma^1{}_{00}(-\Gamma^0{}_{10} + \Gamma^1{}_{11} + \Gamma^2{}_{12})$$

$$= \frac{1}{2}\left(\frac{\kappa Q^2}{2\pi}\right)^2\left[\frac{1}{r^2} - \frac{1}{r^2}\ln^2\left(\frac{r}{r_c}\right) + \frac{1}{r}\ln\left(\frac{r}{r_c}\right)\left(\frac{-1}{2r}\ln\left(\frac{r}{r_c}\right) - \frac{1}{2r}\ln\left(\frac{r}{r_c}\right) + \frac{1}{r}\right)\right]$$

$$= 0$$

$$R_{01} = 0$$

$$R_{02} = 0$$

$$R_{11} = \Gamma^1{}_{11,1} - \Gamma^0{}_{10,1} + \Gamma^1{}_{11,1} + \Gamma^2{}_{12,1} + \Gamma^1{}_{11}(\Gamma^0{}_{10} + \Gamma^1{}_{11} + \Gamma^2{}_{12})$$
$$- \Gamma^0{}_{10}\Gamma^0{}_{10} - \Gamma^1{}_{11}\Gamma^1{}_{11} - \Gamma^2{}_{12}\Gamma^2{}_{12}$$
$$= - \Gamma^0{}_{10,1} + \Gamma^2{}_{12,1} + \Gamma^1{}_{11}(\Gamma^0{}_{10} + \Gamma^2{}_{12}) - \Gamma^0{}_{10}\Gamma^0{}_{10} - \Gamma^2{}_{12}\Gamma^2{}_{12}$$
$$= - \Gamma^0{}_{10,1} + \Gamma^2{}_{12,1} + \Gamma^0{}_{10}(\Gamma^1{}_{11} - \Gamma^0{}_{10}) + \Gamma^2{}_{12}(\Gamma^1{}_{11} - \Gamma^2{}_{12})$$
$$= -\frac{1}{2}\left(-\frac{1}{r^2}\left[\ln\left(\frac{r}{r_c}\right)\right]^{-2} + \left[\ln\left(\frac{r}{r_c}\right)\right]\frac{-1}{r^2}\right) - \frac{-1}{r^2}$$
$$+ \frac{1}{2r}\left[\ln\left(\frac{r}{r_c}\right)\right]^{-1}\left(\frac{-1}{2r}\left[\ln\left(\frac{r}{r_c}\right)\right]^{-1} - \frac{1}{2r}\left[\ln\left(\frac{r}{r_c}\right)\right]^{-1}\right)$$
$$+ \frac{1}{r}\left(\frac{-1}{2r}\left[\ln\left(\frac{r}{r_c}\right)\right]^{-1} - \frac{1}{r}\right)$$
$$= 0$$

$$R_{12} = 0$$

$$R_{22} = \Gamma^1{}_{22,1} + \Gamma^1{}_{22}(\Gamma^0{}_{10} + \Gamma^1{}_{11} + \Gamma^2{}_{12}) - 2\Gamma^2{}_{12}\Gamma^1{}_{22}$$
$$= \Gamma^1{}_{22,1} + \Gamma^1{}_{22}(\Gamma^0{}_{10} + \Gamma^1{}_{11} - \Gamma^2{}_{12})$$
$$= \frac{\kappa Q^2}{2\pi}\Big[1 + \ln\left(\frac{r}{r_c}\right)$$
$$+ r\ln\left(\frac{r}{r_c}\right)\left(\frac{1}{2r}\left[\ln\left(\frac{r}{r_c}\right)\right]^{-1} - \frac{1}{2r}\left[\ln\left(\frac{r}{r_c}\right)\right]^{-1} - \frac{1}{r}\right)\Big]$$
$$= \frac{\kappa Q^2}{2\pi}\left[1 + \ln\left(\frac{r}{r_c}\right) - \ln\left(\frac{r}{r_c}\right)\right]$$
$$= \frac{\kappa Q^2}{2\pi}$$

So all components of R_{ab} are identically zero except for R_{22}. This means that the only non-zero component of $R^a{}_a$ is:

$$R^2{}_2 = g^{22}R_{22} = \frac{1}{r^2}\frac{\kappa Q^2}{2\pi},$$

and the Ricci scalar is:

$$R = R^a{}_a = R^2{}_2 = \frac{\kappa Q^2}{2\pi}\frac{1}{r^2}\,.$$

Of course this could have also been derived from equation (1.10):

$$R = -\kappa^{1}/_{4\pi}(F_{01}F^{01} + F_{10}F^{10})$$
$$= -\kappa^{1}/_{4\pi}(-Q^2/_{r^2} - Q^2/_{r^2})$$
$$= \frac{\kappa Q^2}{2\pi}\frac{1}{r^2},$$

Notes

[1] M. Alpert, *General Relativity in Flatland*, (Astrophysics Undergraduate Thesis: Princeton University, 1982).

Chapter 5

INSIDE-OUT "BLACK HOLE" IN FLATLAND

Let us examine the recall the behavior of the electrostatic metric for r near and greater than r_c. The metric is:

$$ds^2 = \frac{\chi Q^2}{2\pi} \ln\left(\frac{r}{r_c}\right) dt^2 - \frac{2\pi}{\chi Q^2}\left[\ln\left(\frac{r}{r_c}\right)\right]^{-1} dr^2 + r^2 d\varphi$$

as $r \to r_c^-$: $g_{tt} \to 0-$ $g_{rr} \to \infty$

as $r \to r_c^+$: $g_{tt} \to 0+$ $g_{rr} \to -\infty$

as $r \to \infty$: $g_{tt} \to \infty$ $g_{rr} \to 0-$

In his thesis Alpert noted the interesting behavior of the metric near $r=r_c$.[1] He even noticed that this behavior is very similar to the schwarzchild radius of a black hole. He stopped pursuit here, however, claiming that his universe must be closed and have finite mass. To do this he declared r_c the edge of his universe and decided that the universe turned around back on itself at this point. We can continue following the black hole analogy, however, and see that r becomes timelike instead of spacelike, and so continuing along $r>r_c$ would not change the spacelike character of the universe. We will here investigate some of the phenomena associated with $r>r_c$ for the metric.

As we continue along past r_c, r changes from timelike to spacelike, and t changes from spacelike to timelike. Qualitatively, it is the same as what happens at $r = 2GM$ in the schwarzchild metric, except the change occurs going from small r to large r, instead of the other way around. In fact it is even more similar to the case of the black hole in 4 dimensions with charge $Q<\sqrt{G}M$, where there are an inner and outer horizon at r_- and r_+. If we could somehow force $r_+\to\infty$, then r_c would be very much like r_-.

To be certain the black hole analogy holds, we must be sure that the apparent singularity associated with the metric at $r=r_c$ arises purely from the coordinate system, and not from the metric itself. First we must check that the proper distance from $r=0$ to $r=r_c$ is finite. If this distance is infinite, it hardly matters what happens at r_c, since no observer could arrive there in a finite time. To determine the proper distance, let us choose a radial curve from 0 to r_c for which time is constant ($dt = d\varphi = 0$).

$$ds^2 = -\frac{2\pi}{KQ^2}\left[\ \ln\left(\frac{r}{r_c}\right)\right] dr^2$$

$$s(0,r_c) = \int_0^{r_c} ds = \int_0^{r_c} \left(\frac{2\pi}{KQ^2}\right)^{1/2} \left[-\ln\left(\frac{r}{r_c}\right)\right]^{-1/2} dr$$

$$= \int_0^1 \left(\frac{2\pi}{KQ^2}\right)^{1/2} r_c\,[-\ln(\rho)]^{-1/2}\, d\rho$$

$$w = [-\ln(\rho)]^{1/2} \qquad \rho = \exp(-w^2)$$

$$dw = -{}^1/_2\,[-\ln(\rho)]^{-1/2}\,\rho^{-1}\, d\rho = -{}^1/_2\,[-\ln(\rho)]^{-1/2}\exp(w^2)\, d\rho$$

$$s = \int_0^{\infty} \left(\frac{2\pi}{KQ^2}\right)^{1/2} 2\, r_c \exp(-w^2)dw$$

$$s = \left(\frac{2\pi}{KQ^2}\right)^{1/2} 2\, r_c \frac{1}{2}\sqrt{\pi} = \frac{\pi r_c}{Q}\left(\frac{2}{K}\right)^{1/2},$$

which is most definitely finite.

Secondly, we must be sure that the invariant curvature scalars of the metric are non-singular at r_c. From Weinberg[2] we find that there are only 3 invariant curvature scalars associated with the metric in 3 dimensions:

$$R,\ R_{mn}R^{mn},\ \text{and}\ \frac{\det R}{\det g}.$$

We recall from chapter 4 that all $R_{mn} = 0$ except $R_{22} = \kappa Q^2/2\pi$, all $R^{\mu}{}_{\nu} = 0$ except $R^2{}_2 = \kappa Q^2/(2\pi r^2)$, and all $R^{\mu\nu} = 0$ except $R^{22} = \kappa Q^2/(2\pi r^4)$. This gives us the 3 invariant curvature scalars as:

$$R = R^2{}_2 = \kappa Q^2/(2\pi r^2),$$

$$R_{mn}R^{mn} = R_{22}R^{22} = \left(\frac{\kappa Q^2}{2\pi}\right)^2 \frac{1}{r^4}$$

$$\frac{\det R}{\det g} = \frac{0}{-r^2} = 0,$$

none of which show any singularities at r_c, although there is a true singularity in the curvature at r=0.

Having now shown that the apparent singularity at r_c is purely an artifact of the coordinate system (t, r, φ), we should try to find a new coordinate system, analogous to the Kruskal coordinates of a Black Hole, in which the apparent singularity would not appear.[3]

First, for convenience of notation, let us define an effective radius M (with units of length):

$$M \equiv \frac{2\pi r_c}{\kappa Q^2}.$$

The metric can be written:

$$ds^2 = \frac{r_c}{M}\,\ln\left(\frac{r}{r_c}\right) dt^2 - \frac{M}{r_c}\left[\ln\left(\frac{r}{r_c}\right)\right]^{-1} dr^2 + r^2 d\varphi$$

One property we would like is that light beams be orthogonal in this coordinate system. This would be true in a coordinate system of the type:

$$ds^2 \qquad = f(r^*,t)\,(-dt^2 + dr^{*2}) \;+ g(r^*,t)\,d\varphi^2$$

This is easily accomplished by defining

$$dr^* \equiv - \frac{M}{r_c} \frac{dr}{\ln(r/r_c)}$$

so,

$$ds^2 = -r_c/M \ \ln(r/r_c) \ (-dt^2 + dr^{*2}) + r^2 d\varphi^2$$

and r is now an implicit function of r^*, defined by the equation

$$r^* = -M \int_0^r \frac{dr'/r_c}{\ln(r'/r_c)}$$

$$= -M \left[\ \ln \left| \ln\left(\frac{r}{r_c}\right) \right| + \ln\left(\frac{r}{r_c}\right) + \frac{(\ln(r/r_c))^2}{2 \cdot 2!} + \frac{(\ln(r/r_c))^3}{3 \cdot 3!} + \dots \right]$$

$$= -M \ \ln \left| \ln\left(\frac{r}{r_c}\right) F\left(\frac{r}{r_c}\right) \right|,$$

where the function $F(\rho)$ is defined:

$$F(\rho) \equiv \exp\left[\frac{\ln\rho}{1 \cdot 1!} + \frac{(\ln\rho)^2}{2 \cdot 2!} + \frac{(\ln\rho)^3}{3 \cdot 3!} + \dots \right] = \rho \exp\left[\frac{(\ln\rho)^2}{2 \cdot 2!} + \frac{(\ln\rho)^3}{3 \cdot 3!} + \dots \right].$$

Note that $F(\rho)$ is perfectly well behaved, in fact $F(1)=F'(1)=1$ (see appendix B), so there are no singularities being hidden at r_c by this function. This metric is still not a good one at r_c, however, so we must somehow remove the $\ln(r/r_c)$ in front, preferably incorporating it into some new function of r^* & t. We also want the $\ln(r/r_c)$ term to be absorbed equally by both terms so that light rays will remain orthogonal. We find the functions U & V fit the bill.

$$U \equiv \exp\left(\frac{t-r^*}{2M}\right) = \exp\left(\frac{t}{2M}\right) \exp\left(\frac{-r^*}{2M}\right) = \exp\left(\frac{t}{2M}\right) \left(-\ln\left(\frac{r}{r_c}\right) F\left(\frac{r}{r_c}\right)\right)^{1/2}$$

$$V \equiv \exp\left(\frac{-t-r^*}{2M}\right) = \exp\left(\frac{-t}{2M}\right) \exp\left(\frac{-r^*}{2M}\right) = \exp\left(\frac{-t}{2M}\right) \left(-\ln\left(\frac{r}{r_c}\right) F\left(\frac{r}{r_c}\right)\right)^{1/2}$$

Note that these definitions only apply for $r<r_c$, where

$$\left| \ln\left(\frac{r}{r_c}\right) \right| = - \ln\left(\frac{r}{r_c}\right),$$

so for the time being we will confine ourselves to this side of r_c.

$$UV = -F\left(\frac{r}{r_c}\right) \ln\left(\frac{r}{r_c}\right) = \exp\left(\frac{-r^*}{M}\right)$$

so,

$$ds^2 = \frac{r_c}{M}\,\frac{UV}{F(r/r_c)}\,(-dt^2+dr^{*2}) + r^2 d\varphi^2$$

and,

$$dU = \frac{1}{2M}\exp\left(\frac{t-r^*}{2M}\right)\left(dt-dr^*\right)$$

$$dV = \frac{1}{2M}\exp\left(\frac{-t-r^*}{2M}\right)\left(-dt-dr^*\right).$$

which gives us

$$(-dt^2+dr^{*2}) = 4M^2\exp\left(\frac{r^*}{M}\right) dU\,dV$$

$$ds^2 = 4M\,r_c\,\frac{UV}{F(r/r_c)}\exp\left(\frac{r^*}{M}\right) dU\,dV + r^2 d\varphi^2,$$

and we end up with,

$$ds^2 = \frac{4M\,r_c}{F(r/r_c)}\,dU\,dV + r^2 d\varphi^2,$$

where r is a function of U & V.

This as it stands is a perfectly valid coordinate system, but U & V are null coordinates; we can easily replace them with more familiar timelike and spacelike coordinates:

$$u = \frac{1}{2}(U+V) = \exp\left(\frac{-r^*}{2M}\right)\cosh\left(\frac{t}{2M}\right) = \left(-\ln\left(\frac{r}{r_c}\right) F\left(\frac{r}{r_c}\right)\right)^{1/2}\cosh\left(\frac{t}{2M}\right)$$

$$v = \frac{1}{2}(U-V) = \exp\left(\frac{-r^*}{2M}\right)\sinh\left(\frac{t}{2M}\right) = \left(-\ln\left(\frac{r}{r_c}\right) F\left(\frac{r}{r_c}\right)\right)^{1/2}\sinh\left(\frac{t}{2M}\right),$$

which gives us

$$du = {}^1/_2\,(dU+dV)$$

$$dv = {}^1/_2\,(dU-dV),$$

so

$$-dv^2 + du^2 = dUdV$$

$$ds^2 = \frac{4M\, r_c}{F(r/r_c)} \; (-dv^2 + du^2) + r^2 d\varphi^2 \; .$$

Recalling the definition of M, we get for our metric:

$$ds^2 = \frac{8\pi r_c^2}{\varkappa Q^2 F(r/r_c)} (-dv^2 + du^2) + r^2 d\varphi^2 .$$

This is the metric in good coordinates for $r<r_c$. Notice that r is now a function of u & v, defined implicitly by the equation

$$u^2 - v^2 = -F(r/r_c) \; \ln(r/r_c).$$

(We can calculate t from u & v by noticing that

$$v/u = \tanh(t/2M)$$

$$t \quad = \frac{4\pi r_c}{\varkappa Q^2} \tanh^{-1}\left(\frac{v}{u}\right) \quad .)$$

The same procedure can be applied in the case of $r>r_c$, with suitable changes in the definitions of U, V, u, & v, remembering that for $r>r_c$

$$\left| \ln(r/r_c) \right| = \ln(r/r_c).$$

Following the same steps as above:

$$U \;\equiv\; \exp\left(\frac{t-r^*}{2M}\right) = \exp\left(\frac{t}{2M}\right) \exp\left(\frac{-r^*}{2M}\right) \;=\; \exp\left(\frac{t}{2M}\right) \left(\ln\left(\frac{r}{r_c}\right) \; F\left(\frac{r}{r_c}\right) \right)^{1/2}$$

$$V \;\equiv -\exp\left(\frac{-t-r^*}{2M}\right) = -\exp\left(\frac{-t}{2M}\right) \exp\left(\frac{-r^*}{2M}\right) = -\exp\left(\frac{-t}{2M}\right) \; \left(\ln\left(\frac{r}{r_c}\right) \; F\left(\frac{r}{r_c}\right) \right)^{1/2}$$

$$UV = \; -F\left(\frac{r}{r_c}\right) \; \ln\left(\frac{r}{r_c}\right) \; = -\exp\left(\frac{-r^*}{M}\right)$$

$$ds^2 \;=\; \frac{r_c}{M} \; \frac{UV}{F(r/r_c)} (-dt^2 + dr^{*2}) + r^2 d\varphi^2$$

$$dU \;\; = \frac{1}{2M} \exp\left(\frac{t-r^*}{2M}\right) \; (\, dt - dr^*)$$

$$dV \;\; = \frac{-1}{2M} \; \exp\left(\frac{-t-r^*}{2M}\right) \; (-dt - dr^*)$$

$$(-dt^2 + dr^{*2}) = -\, 4M^2 \; \exp\left(\frac{r^*}{M}\right) \; dU\, dV$$

$$ds^2 = 4M\,r_c \frac{UV}{F(r/r_c)} \exp\left(\frac{r^*}{M}\right) dU\,dV + r^2 d\varphi^2 = \frac{4M\,r_c}{F(r/r_c)} dU\,dV + r^2 d\varphi^2$$

$$u = \frac{1}{2}(U+V) = \exp\left(\frac{-r^*}{2M}\right) \sinh\left(\frac{t}{2M}\right) = \left(\ln\left(\frac{r}{r_c}\right) F\left(\frac{r}{r_c}\right)\right)^{1/2} \sinh\left(\frac{t}{2M}\right)$$

$$v = \frac{1}{2}(U-V) = \exp\left(\frac{-r^*}{2M}\right) \cosh\left(\frac{t}{2M}\right) = \left(\ln\left(\frac{r}{r_c}\right) F\left(\frac{r}{r_c}\right)\right)^{1/2} \cosh\left(\frac{t}{2M}\right)$$

$$du = {}^1/_2\,(dU+dV)$$

$$dv = {}^1/_2\,(dU-dV),$$

$$-dv^2 + du^2 = dUdV$$

$$ds^2 = \frac{4M\,r_c}{F(r/r_c)} (-dv^2 + du^2) + r^2 d\varphi^2 .$$

$$ds^2 = \frac{8\pi r_c^2}{\kappa Q^2 F(r/r_c)}(-dv^2 + du^2) + r^2 d\varphi^2$$

Note that this metric is the same as above for the case that $r<r_c$. Note also that $u^2-v^2=-F(r/r_c)\ln(r/r_c)$ is still true, regardless of whether $r<r_c$ or $r>r_c$. (In this region we get a different equation for t in terms of u & v:

$$t = \frac{4\pi r_c}{\kappa Q^2} \tanh^{-1}\left(\frac{u}{v}\right) .)$$

Next we try to show that the metric in terms of u & v is complete. We do this by showing that the total proper time between $v=0$ and $v=\infty$ for $u=0$ (which is the same as the total proper time between $r=r_c$ and $r=\infty$ for $t=0$), is infinite.

$$ds^2 = -\frac{2\pi}{\kappa Q^2}\left[\ln\left(\frac{r}{r_c}\right)\right]^{-1} dr^2 \text{ for } dt=d\varphi=0$$

$$\tau(r_c,\infty) = \int_{r_c}^{\infty} d\tau = \int_{r_c}^{\infty} \left(\frac{2\pi}{\kappa Q^2}\right)^{1/2} \left[\ln\left(\frac{r}{r_c}\right)\right]^{-1/2} dr$$

$$= \int_1^{\infty} \left(\frac{2\pi}{\kappa Q^2}\right)^{1/2} r_c\,[\ln(\rho)]^{-1/2}\, d\rho$$

$$w = [\ln(\rho)]^{1/2} \qquad \rho = \exp(w^2)$$

$$dw = {}^1/_2\,[\ln(\rho)]^{-1/2} \rho^{-1}\, d\rho = {}^1/_2\,[\ln(\rho)]^{-1/2} \exp(-w^2)\, d\rho$$

$$\tau = \int_0^\infty \left(\frac{2\pi}{\kappa Q^2}\right)^{1/2} 2\, r_c \exp(w^2)dw = \infty$$

Since the proper time elapsed from v=0 to v=∞ is infinite, we can conclude that this coordinate system is complete. (Actually, there is one direction in which we have not proven that the metric is complete, and that is along null paths of (u ± v) = constant. This calculation is very difficult, however, and it will not be attempted here.)

Now we can see how various points and curves transform from the (r,t) plane onto the (u,v) plane in diagram 5.1. The line t=0 corresponds to the line v=0. For $r=r_c$ we get $u^2=v^2$, which are the two lines u=v and u=-v.

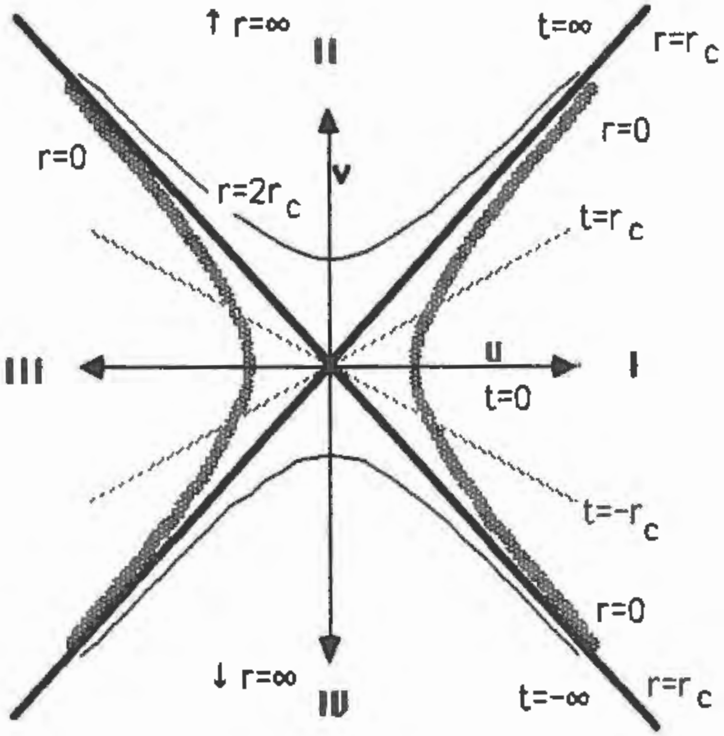


Diagram 5.1

The region $r<r_c$ is the same as $u^2>v^2$, which is represented by regions I and III. The region $r>r_c$ corresponds to $u^2<v^2$, which is represented by regions II and IV. We know from appendix B that when r=0, $F(r/r_c)\ln(r/r_c)=-1$, so $u^2-v^2=1$, shown as the two shaded hyperbolas. There is a true singularity here, so any points beyond these curves can not be interpreted within General Relativity. The other curves follow easily from the definitions of u & v.

There are obvious similarities with the schwarzchild case with the u-v plane turned on its side. From this diagram we can see that the horizon at $r=r_c$ is not a true event horizon, but, as noted above, is qualitatively very similar to the inner horizon, r_-, of a charged black hole in four dimensions. The most important feature of this horizon is that the singularity "behind" it is avoidable along timelike geodesics.

The following two diagrams are embedding diagrams of this space in a higher dimensional space. The first is along the hypersurface t=0, from one r=0, to $r=r_c$, and back to the other r=0. As the reader can see, this is the same imbedding diagram as (4.3).

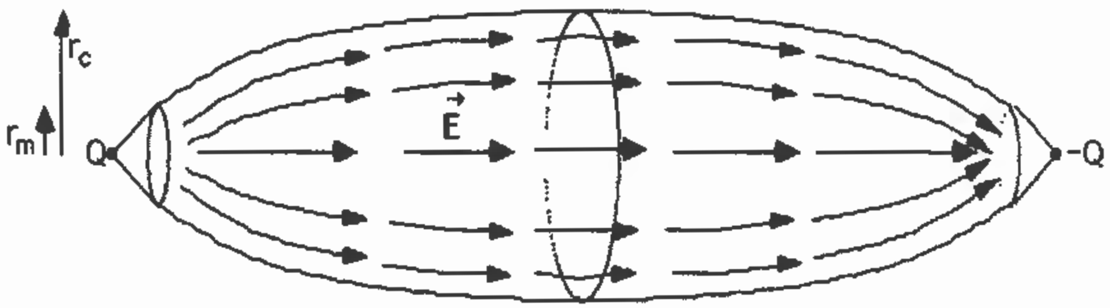


Diagram 5.2

Diagram 5.3 covers a part of space-time which is not dealt with at all in chapter 4. This is one of the hypersurfaces $r>r_c$, which are shown as hyperbolas in diagram 5.1. Along this hypersurface, the electric field lines are constant, and the field lines arise from charges that are infinitely far away (Gauss's law is still satisfied). Remember, however, that in this section of the universe r is timelike; it is t that is spacelike.

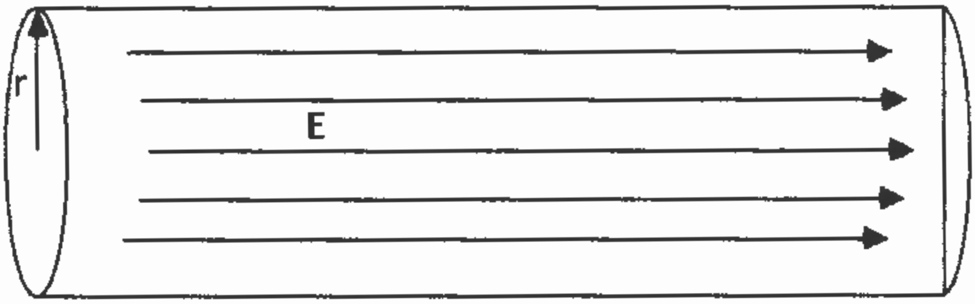


Diagram 5.3

As time progresses, r increases, and we can see that diagram 5.1 represents an evolving cosmology, much more than just the static universe Alpert had surmised.[1] This universe does not expand isotropically, but with a cylindrical symmetry (the Hubble constant is dependent on direction).

On another note the reader should take heed that, while this cosmology behaves perfectly well within the realm of General Relativity, there may still remain problems when quantum mechanical effects are brought in. The primary question in this area is whether this universe is stable against pair-particle creation in high electric fields as $r \to 0$; if it is unstable, region II of the universe might not form, and the universe would end in a finite proper time once r_c is crossed.

Notes

[1] M. Alpert, *General Relativity in Flatland*, (Astrophysics Undergraduate Thesis: Princeton University, 1982).

[2] S. Weinberg, *Gravitation and Cosmology*, (Wiley and Sons: New York, 1972).

[3] The methods used in this section are adapted from those used in: C.W. Misner, K.S. Thorne, and J.A. Wheeler, *Gravitation*, (W.H. Freeman: San Francisco, 1973).

Appendix A: Cyclic Coordinates

Conservation of covariant momentum along a geodesic for cyclic coordinates is an important tool in solving for particle motion in General Relativity. Yet a straightforward proof of this that does not use the action principle is dificult to find in the literature. Below is a fairly simple proof, modified from Lightman et al. (see Bibliography).

Given:

1) $u^\alpha = \frac{dx^\alpha}{d\lambda}$ 2) $g_{\alpha\beta,3} = 0$

0	$= \frac{du^\mu}{d\lambda} + \Gamma^\mu{}_{\alpha\beta} u^\alpha u^\beta$	Geodesic Equation
	$= g_{\delta\mu}\frac{du^\mu}{d\lambda} + \Gamma_{\delta\alpha\beta} u^\alpha u^\beta$	$\times\ g_{\delta\mu}$
	$= g_{\delta\alpha}\frac{du^\alpha}{d\lambda} + \Gamma_{\delta\alpha\beta} u^\alpha u^\beta$	change dummy indices ($\mu \to \alpha$)
	$= \frac{d}{d\lambda}(g_{\delta\alpha}u^\alpha) - (\frac{d}{d\lambda} g_{\delta\alpha})u^\alpha + \Gamma_{\delta\alpha\beta} u^\alpha u^\beta$	product rule
	$= \frac{du_\delta}{d\lambda} - g_{\delta\alpha,\beta}\frac{dx^\beta}{d\lambda}u^\alpha + \Gamma_{\delta\alpha\beta} u^\alpha u^\beta$	chain rule
	$= \frac{du_\delta}{d\lambda} - g_{\delta\alpha,\beta}u^\beta u^\alpha + \Gamma_{\delta\alpha\beta} u^\alpha u^\beta$	
	$= \frac{du_\delta}{d\lambda} - [g_{\delta\alpha;\beta} + \Gamma^\mu{}_{\alpha\beta} g_{\delta\mu} + \Gamma^\mu{}_{\delta\beta} g_{\alpha\mu}]u^\alpha u^\beta + \Gamma_{\delta\alpha\beta} u^\alpha u^\beta$	semicolon rule for tensors
	$= \frac{du_\delta}{d\lambda} - [\Gamma_{\delta\alpha\beta} + \Gamma_{\alpha\delta\beta} - \Gamma_{\delta\alpha\beta}]u^\alpha u^\beta$	
	$= \frac{du_\delta}{d\lambda} - \Gamma_{\alpha\beta\delta}u^\alpha u^\beta$	

$\frac{du_3}{d\lambda} = \Gamma_{\alpha\beta 3}\, u^\alpha u^\beta$ 3-component by above rule

$= \Gamma_{(\alpha\beta)3}u^\alpha u^\beta$ switch dummy indices and symmetrize

$= {}^1/_2\, g_{\alpha\beta,3}\, u^\alpha u^\beta$ symmetric part of Γ, from definition

$= 0$ $\Rightarrow$ u_3 = constant

Appendix B: Properties of $F(\rho)$

The function $F(\rho)$ is defined above as:

$$F(\rho) \equiv \exp\left(\frac{\ell n\rho}{1\cdot 1!} + \frac{(\ell n\rho)^2}{2\cdot 2!} + \frac{(\ell n\rho)^3}{3\cdot 3!} + \ldots \right).$$

The derivative is given by:

$$F'(\rho) = F(\rho)\left[\left(\frac{1}{\rho}\right) \left(\frac{(\ell n\rho)^0}{1!} + \frac{(\ell n\rho)^1}{2!} + \frac{(\ell n\rho)^2}{3!} + \ldots \right) \right]$$

$$= F(\rho)\left[\left(\frac{1}{\rho}\right) \left(\frac{1}{\ell n\rho}\right) \left(\frac{(\ell n\rho)^1}{1!} + \frac{(\ell n\rho)^2}{2!} + \frac{(\ell n\rho)^3}{3!} + \ldots \right) \right]$$

$$= F(\rho)\left[\left(\frac{1}{\rho\,\ell n\rho}\right) \left(-1 + 1 + \frac{(\ell n\rho)^1}{1!} + \frac{(\ell n\rho)^2}{2!} + \frac{(\ell n\rho)^3}{3!} + \ldots \right) \right]$$

$$= F(\rho)\left[\left(\frac{1}{\rho\,\ell n\rho}\right) \left(-1 + \exp(\ell n\rho) \right) \right]$$

$$F'(\rho) = F(\rho)\left[\frac{\rho - 1}{\rho\,\ell n\rho} \right]$$

It is easily seen that any interesting behavior in $F(\rho)$ or $F'(\rho)$ will be at $\rho=0$ or $\rho=1$. From the definition of $F(\rho)$ it is seen that $F(1) = 1$, and the derivative is given in the limit:

$$F'(1) = \lim_{\rho\to 1} F(\rho)\left[\frac{\rho-1}{\rho\,\ell n\rho} \right]$$

$$= \lim_{\rho\to 1} \frac{\rho-1}{\rho\,\ell n\rho}$$

$$= \lim_{\rho\to 1} \frac{1}{1+\ell n\rho} \qquad \text{by L'Hopital's Rule}$$

$$F'(1) = 1$$

Evaluating $F(0)$ is a little trickier since it is unclear from the definition how the function behaves in this limit. First we must notice how $F(\rho)$ fits into the definition of r^*.

$$r^* = -M\,\ell n\left| \ell n\left(\frac{r}{r_c}\right) F\left(\frac{r}{r_c}\right) \right|.$$

Since r^* is defined as an integral from 0 to r, we know that $r^*=0$ when $r=0$. This gives us:

$$0 = \lim_{\rho\to 0} -M \ln\left|\ln(\rho)F(\rho)\right|, \qquad \text{or,}$$

$$1 = \lim_{\rho\to 0} \left|\ln(\rho)F(\rho)\right|.$$

Since $\ln(0) = -\infty$, in order for this quantity to remain finite we must have:

$$F(0)=0$$

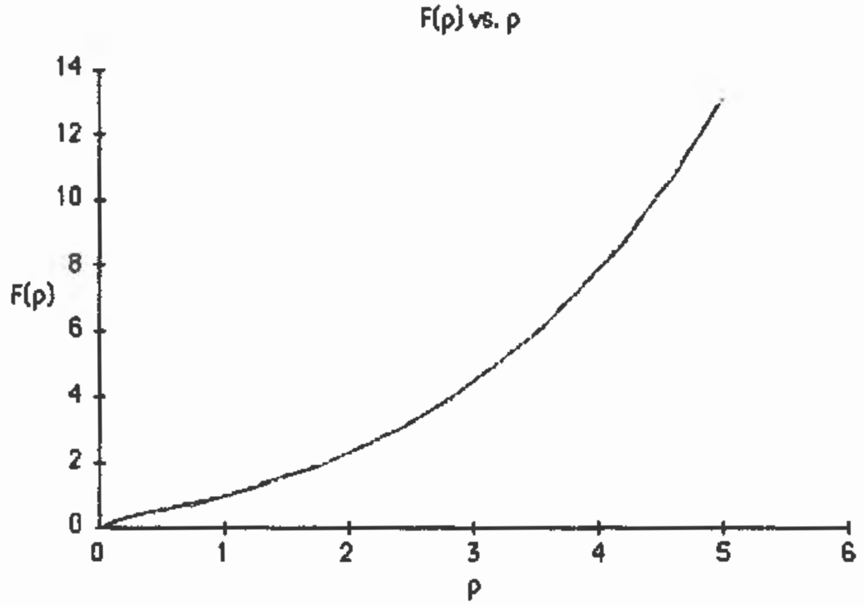
And we can easily calculate $F'(0)$:

$$F'(0)= \lim_{\rho\to 0} F(\rho)\left[\frac{\rho-1}{\rho\ln\rho}\right]$$

$$F'(0)= 0. \qquad \text{since} \lim_{\rho\to 0} \rho\ln\rho = 1$$

The following graph was evaluated using the approximation:

$$F(\rho)\simeq \exp\left[\frac{\ln\rho}{1\cdot 1!} + \frac{(\ln\rho)^2}{2\cdot 2!} + \frac{(\ln\rho)^3}{3\cdot 3!} + \frac{(\ln\rho)^4}{4\cdot 4!} + \frac{(\ln\rho)^5}{5\cdot 5!}\right].$$



(Calculating $F''(\rho)$ reveals points of inflection at $\rho=1$ and $\rho\simeq.3$, and these points can be observed as twists in the graph.)

Bibliography

M. Alpert, *General Relativity in Flatland*, (Astrophysics Undergraduate Thesis: Princeton University, 1982).

S. Deser, R. Jackiw, and G. t'Hooft, Ann. Phys. **152**, 220 (1984)

S. Giddings, J. Abbot, and K. Kuchar, G.R.G. **16**, 751 (1984).

J. R. Gott III and M. Alpert, G.R.G. **16**, 243 (1984).

T. Kaluza, Sitzungsberichte, Preussische Akadamie der Wissenschaften, 966 (1921).

O. Klein, Z. Phys., **37**, 895 (1926).

L.D. Landau and E.M. Lifshitz, *The Classical Theory of Fields*, (Addison Wesley: Reading, MA 1962).

A.P. Lightman, W.H. Press, R.H. Price, and S.A. Teukolosky, *Problem Book in Relativity and Gravitation*, (Princeton University Press: Princeton NJ, 1979).

C.W. Misner, K.S. Thorne, and J.A. Wheeler, *Gravitation*, (W.H. Freeman: San Francisco, 1973).

S. Weinberg, *Gravitation and Cosmology*, (Wiley and Sons: New York, 1972).