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Higher Derivative Expansions and Non-Locality

with Applications to Gravity and the Stability of Flat Space

A Dissertation submitted in partial satisfaction of the requirements for the degree of

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by

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This work is dedicated to my mother,
Nancy Simon, whom I miss very much.

Curriculum Vitae

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1983 - 1984	Teaching High School physics at Exploration Summer Program, Wellesley College
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"I am not getting anything out of the meeting. I am learning nothing. Because there are no experiments this field is not an active one, so few of the best men [sic] *are doing work in it. ... It is not that the subject is hard; it is that the good men are occupied elsewhere. Remind me not to come to any more gravity conferences!"*

Richard P. Feynman,

with whom I occasionally must disagree

Abstract

Higher derivative theories are frequently avoided because of undesirable properties, yet they occur naturally as corrections to general relativity and cosmic strings. We discuss some of their more interesting and disturbing problems, with examples. A natural method of removing all the problems of higher derivatives is reviewed. This method of "perturbative constraints" is required for at least one class of higher derivative theories, those which are associated with non-locality. Non-locality often appears in low energy theories described by effective actions. The method may also be applied to a wide class of other higher derivative theories. An example system is solved, exactly and perturbatively, for which the perturbative solutions approximate the exact solutions only when the method of "perturbative constraints" is employed. Ramifications for corrections to general relativity, cosmic strings with rigidity terms, and other higher derivative theories are explored.

Next, flat space is shown to be perturbatively stable, to first order in $\hbar$, against quantum fluctuations produced in semiclassical (or $1/N$ expansion) approximations to quantum gravity, despite past indications to the contrary. It is pointed out that most of the new "solutions" allowed by the semiclassical corrections do not fall within the perturbative framework, unlike the effective action and field equations which generate them. It is shown that excluding these non-perturbative "pseudo-solutions" is the only self-consistent approach. The remaining physical solutions do fall within the perturbative formalism, do not require the introduction of new degrees of freedom, and suffer none of the pathologies of unconstrained higher derivative systems. The presence of the higher derivative terms in the semiclassical corrections may be related to non-locality.

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Chapter 1

Introduction

Descriptions of physical systems that require differential equations of degree higher than two in time derivatives are uncommon in physics. Often it is taken for granted that only the initial position and velocity of an object are necessary to determine its trajectory. Newton's second law can be interpreted as an example of this principle.

$$\ddot{\mathbf{x}} = m^{-1}\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}) \tag{1 - 1}$$

i.e. the second time derivative is an explicit function of the first and zeroth derivatives. There are some exceptions to this rule of thumb, however, where small corrections can be higher than first order in time. The oldest examples of this are corrections to the equation of motion for charged particles which take into account radiative effects. The Abraham-Lorentz equation is (see e.g. Jackson[1])

$$\ddot{\mathbf{x}} = m^{-1}\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}) + \varepsilon\dddot{\mathbf{x}} \tag{1 - 2}$$

where $\varepsilon = 2e^2/(3mc^3)$. This predicts that the power lost to radiation is given by the Larmor formula

$$P_{\text{lost}} = \frac{2e^2}{3c^3}\ddot{\mathbf{x}}^2 \tag{1 - 3}$$

which is consistent with the energy loss predicted by electromagnetic field theory. For the electron, where $\varepsilon = 6.26 \times 10^{-24}$ sec is quite a small time scale, this would ordinarily be a small correction when accelerations are small. On the other hand, there is a fundamental difference between equation (1 - 2) and equation (1 - 1) since the differential equation is now third order. If (1 - 2) is correct and complete as it stands, the initial position, velocity, and acceleration must all be specified to evolve the system forward in time.

This thesis classifies higher derivative theories and describes methods for augmenting their field equations with constraints which eliminate solutions inappropriate to the theory being described. Most physical theories are described by a second order differential equation and (all of) its solutions. At least some theories, however, are described by a higher order differential equation and *some* of its solutions. Higher derivative theories can roughly be classified into three cases:

1) Theories such as the electrodynamics described above (and also the relativistic generalization by Dirac[2]), where the differential equations do not completely define the theory, but must be supplemented with constraints or boundary conditions to eliminate solutions not belonging to the theory

2) Theories in which the higher derivatives arise in the process of constructing a perturbative approximation to a more fundamental theory. Constraints are needed to eliminate spurious solutions to the higher order differential equations that are not approximations to the solutions of the underlying equations. Examples of this are perturbative expansion approximations to non-local theories, as will be described below. The distinction between theories of the first and second category can be blurry if the constraints used to remove spurious solutions from both types are the same, if there is more than one derivation of a given theory, one falling into the first category and another into the second.

3) Genuinely higher order theories, where all solutions to the higher order differential equation correspond to physical trajectories. We know of no physical examples, but there is no fundamental reason they should not occur either. These theories do have have somewhat strange behavior not found in ordinary second order theories, including negative kinetic energy, but they are mathematically self-consistent.

For classical electrodynamics, as above and in Dirac's generalization, the higher derivative terms arise from the particle's interaction with radiation emitted by the same particle, e.g. an electron interacting with its own bremsstrahlung. Depending on the particular derivation (there are several), it can fall into either the first or second category.

Higher derivative corrections to general relativity arise from interactions with quantum matter fields, or can be posited a priori to make the field theory renormalizable, and also arise from superstring corrections. The quantum corrections are particularly important in the context of quantum field theory in curved spacetime. Important physical situations described by these quantum corrections include the back reaction of Hawking radiation on the metric of a large evaporating black hole, and the back reaction of particles created in the cosmological transition from an inflationary era to a radiation dominated era.

Depending on the derivation of these corrections, the higher derivative theory could conceivably fall into any of the three categories (though corrections due to superstrings fall into the second category only).

Higher derivative corrections to the field equations of cosmic strings occur as terms proportional to the curvature of the cosmic strings. Cosmic strings are potentially important in the formation of galaxies. They could occur in a universe undergoing cooling with spontaneous symmetry breaking. They are line defects in a scalar gauge field theory in the same way that magnetic monopoles are point defects, contained for topological reasons. When the strings are mostly straight (the radius of curvature is much larger then the width of the string), their behavior is governed by the Nambu-Goto action, equal to the area of the world-sheet swept out in time (this is the 2 dimensional generalization of the action of a free relativistic particle, which is equal to the length of the world-line). When the curvature increases, however, as it does near kinks and cusps formed when strings intersect, higher order terms would be expected to grow in importance. The first order corrections, higher derivative in the world sheet coordinates, are generated by truncating an infinite expansion in powers of the string width, and so fall into the second category of higher order theories.

Higher derivatives can have disastrous consequences if all the solutions to the higher order equations of motion are taken to be physical (i.e. predictions of the theory). Take for example the Abraham-Lorentz model above, in the case of zero external force ($\mathbf{F} = 0$). The exact solution is

$$\mathbf{x} = (\mathbf{x}_0 - \varepsilon^2 \mathbf{a}_0) + (\mathbf{v}_0 - \varepsilon \mathbf{a}_0)t + \varepsilon^2 \mathbf{a}_0 \exp(t / \varepsilon). \qquad (1 - 4)$$

If all solutions to this equation are considered, the prediction would be that for zero initial position and velocity, and no external forces, any non-zero initial acceleration, no matter how small, would give exponential acceleration, with an e-folding time of 10^{-23} sec. This is an example of a typical byproduct of higher derivative corrections, often called "runaway solutions". In the case of the Abraham-Lorentz and Dirac models of charged particles, this is obviously an undesirable prediction. Hence, it has always been recognized that the higher derivative equation of motion alone is not a good description of the physical system, and that some supplementary rule for excluding undesirable solutions is necessary. Historically, Dirac and others used a future boundary condition, that, at late times, after all external forces have died off, the final acceleration should be zero. This supplementary condition not only removes the runaway solutions, but no longer can the initial acceleration be specified independently of the initial position and velocity (though now we have given up an initial value formulation by relying on future behavior).

This thesis puts forward a method of restricting solutions to the higher order equations of motion that is different and more general than Dirac's (although they agree in some simple cases). While Dirac's method is acceptable if the extra solutions do "run away" in some sense, this does not always occur in an obvious way. It is not always straightforward to distinguish between "obviously" physical solutions and pathological solutions. A simple model presented below, for instance, has extra solutions which are always bounded, but oscillate on an extremely rapid time scale and with negative energy. Standard general relativity has many bizarre solutions even without higher derivative corrections, and distinguishing between "obviously physical" solutions and "pathological" solutions arising from the higher derivative quantum corrections is not a well defined task.

(Of course the final deciding factor of what is physical and what is not will always be experiment, but this cannot help us yet. While experiment tells us that obviously pathological solutions to the Abraham-Lorentz model must be excluded, it cannot help us determine which way it must be done, because, in practice, quantum effects dominate at that scale. Corrections to general relativity are also far too small to be measured at the present (they are on the order of the Planck scale), and measuring corrections to the behavior of cosmic strings will at least need to wait until it is verified that they exist.)

The method put forward here can be used whenever the higher derivative terms in the equation of motion occur as small corrections to a second order theory. More specifically, it applies in any cases where the higher derivative corrections occur only in terms multiplied by a small perturbative expansion parameter. Equation

(1 - 2) falls into this category, since the higher derivative is multiplied by ε, as would any equation of motion in the form

$$\ddot{\mathbf{x}} = m^{-1}\mathbf{F}(\dot{\mathbf{x}},\mathbf{x}) + \sum_{n=1}^{N} c_n \varepsilon^n \mathbf{f}_n\left(\frac{d^n\mathbf{x}}{dt^n}, \frac{d^{n-1}\mathbf{x}}{dt^{n-1}}, \ldots \dot{\mathbf{x}}, \mathbf{x}\right) \tag{1 - 5}$$

for some N. Most solutions, however, to a differential equation of this type do not have perturbative expansions themselves, even though their equations of motion do. For example, equation (1 - 2) is a first order perturbative expansion in ε, but a solution to it, equation (1 - 4), is, in general, not (in particular, the last term has no non-trivial Taylor expansion in ε about $\varepsilon = 0$). The method put forward here, in its simplest form, predicts that only solutions perturbatively expandable in the small perturbative parameter, e.g. ε, correspond to predictions ("physical" solutions), and all other solutions are spurious and do not correspond to anything ("pseudo-solutions").

The most important use of this method (called the method of "perturbative constraints") is when applied to higher derivative systems that are truncated perturbative expansions of non-local systems. Non-local systems are systems for which the equation of motion depends on more than one moment in time. Non-local terms expanded in a Taylor series can become higher derivative expansions, as in the example

$$x(t + \varepsilon T) = \sum_{n=0}^{\infty} \frac{(\varepsilon T)^n}{n!} \frac{d^n}{dt^n} x(t). \tag{1 - 6}$$

In this way, a non-local expression can be put into the form of an infinite sum of (individually local) higher derivative terms. Non-local theories, while commonly perceived as exotic, are not as rare as often believed. A classic example is electrodynamics formulated such that charged particles interact via retarded potentials. There the forces on a particle depend on its own and other particles' positions at times in the past. This is a simple form of non-locality. In many cases, such as this, the underlying theory is local (electromagnetic radiation interacts locally and then propagates), but non-locality can appear in the derived theory nevertheless. The higher derivative corrections to cosmic strings appear because of an expansion of a non-local, derived theory (the underlying gauge field theory is local, but the intermediate derived theory is not). Superstring theories give higher derivative corrections to general relativity by the same mechanism.

Using the method of perturbative constraints is crucial in the treatment of theories for which the higher derivatives appear as a result of perturbatively expanding non-local terms. A simple model is used below to demonstrate. The model is non-local but can be solved, both exactly and perturbatively, and both classically and quantum mechanically. It will be demonstrated below that even in this simple case, pseudo-solutions (solutions not perturbatively expandable in the small parameter) arise spontaneously and must be excluded by the method of perturbative constraints. If the method of perturbative constraints is not applied, the predictions of the higher derivative expansions are not at all like the predictions of the full non-local theory. The reason the higher derivative terms appear at all is because of non-locality, not because they represent new dynamics.

For higher derivative quantum corrections to gravity, it is not clear whether the higher derivatives signify new dynamics, non-locality, or something else. The modified Einstein equations are fourth order instead of second. If the higher derivatives signify new dynamics (i.e. all solutions to the higher order field equations are predictions), then the modified theory is very different from the unmodified theory. Not all the new solutions are "runaways" in the sense of the Abraham-Lorentz and Dirac models, but their behavior is different enough from the Einstein solutions to destabilize flat space, allow negative energy modes, and produce large amounts of Planck-frequency radiation.[3-5] Classically flat space is stable against small perturbations. Small gravitational waves do not interact in such a way to grow without bound. The fourth order modified Einstein equation, however, allows perturbations that do grow without bound, and typically with a time scale of order the Planck time. These modes can grow without bound because their energy is proportional to their amplitude squared with a *negative* proportionality constant (a feature common in unconstrained higher derivative theories). There also exist other negative energy modes that are oscillatory with Planck scale frequencies. These modes would couple to other matter such as electromagnetism, producing extremely high energy photons. All these properties are clearly in conflict with our everyday experience.

Treating the quantum corrections as the first term in a longer perturbative expansion, however, allows us to use the method of perturbative constraints. Excluding the pseudo-solutions in this way gives a theory which predicts that flat space is stable, and the positive energy theorem still holds. This is in accord with our experience, and thus using the perturbative constraints is a much more serious candidate for interpreting the quantum corrections than no constraints at all.

There are still important problems that need to be addressed in the future. The analysis of the stability of flat space could straightforwardly be extended to higher orders. The exact form of the next higher order corrections is not known, but there are a finite number of terms $O(\hbar^2)$ that are local (in the field equations) and have the correct dimensions of $(\text{length})^{-6}$. Perhaps an even more pressing need is a treatment of first order corrections to gravity in cosmological contexts (not near flat space). There, the first order corrections have been used extensively in theoretical models to drive cosmological expansion (in both inflation-like and Robertson-Walker-like solutions). These models may not be consistent with the perturbative origin of the corrections.

The structure of the dissertation is as follows. Chapter 2 is a discussion of the properties of higher derivative theories and non-locality. First is a review of the behavior of unconstrained higher derivative theories in general, both classical and quantum, with simple examples of all interesting properties. Next is a discussion of various higher derivative theories that have been studied in the literature, including Dirac's classical electrodynamics. Non-local, higher derivative corrections to general relativity, and corrections to cosmic strings are analyzed using the necessary perturbative constraints.

For gravity, typical corrections take the form of curvature squared terms in the action. Even for small coefficients (in fact, especially for small coefficients) these terms can dominate the evolution of the system (this is what drives "Starobinsky inflation"). Applying the appropriate perturbative constraints describes a system in which the number of degrees of freedom are the same as in Einstein gravity, and which has no runaway solutions or ghostlike particles (in contrast to the unconstrained system). It is a perturbative correction to Einstein gravity, which we know to be a very good approximation of nature. For gravity as a low energy limit to string theory it is found that, to first order in the slope parameter and for zero matter fields, the string corrections have no effect on Einstein gravity. For the system of cosmic strings with higher derivative corrections (often referred to as rigidity) it is found, to first order in the thickness of the string, the higher order terms have no effect on the behavior of the string.

Chapter 3 applies the method of perturbative constraints to the problem of the stability of flat space. When quantum corrections to gravity coupled with matter are calculated, it is found that they are of the same form as discussed above: the corrections are higher order in time derivatives than the original Einstein equations, with a small coefficient ($\hbar$). These corrections can be calculated by several means, including semiclassical methods and $1/N$ approximations. If the perturbative constraints are not imposed, the system behaves very differently from Einstein gravity, as might be expected: there are more degrees of freedom, and there is no positive energy theorem. Flat space appears to be unstable. When the correct perturbative constraints are imposed, however, the system regains the same properties as Einstein gravity and flat space shows no signs of instability.

Chapter 2

Higher Derivative Lagrangians, Non-Locality, Problems, and Solutions

Higher derivative theories are frequently avoided because of undesirable properties, yet they occur naturally as corrections to general relativity and cosmic strings. We discuss some of their more interesting and disturbing problems, with examples. A natural method of removing all the problems of higher derivatives is reviewed. This method of "perturbative constraints" is required for at least one class of higher derivative theories, those which are associated with non-locality. Non-locality often appears in low energy theories described by effective actions. The method may also be applied to a wide class of other higher derivative theories. An example system is solved, exactly and perturbatively, for which the perturbative solutions approximate the exact solutions only when the method of "perturbative constraints" is employed. Ramifications for corrections to general relativity, cosmic strings with rigidity terms, and other higher derivative theories are explored.

I. Introduction

Theories with higher derivatives (third derivative or higher in time in the equations of motion, second derivative or higher in the Lagrangian) occur naturally for various reasons in different areas of physics. Quite often the higher derivative terms are added to a more standard (lower derivative) theory as a correction. This occurs in general relativity, for instance, where quantum corrections naturally contain higher derivatives of the metric (see *e.g.* Birrell and Davies[6]), or where non-linear sigma models of string theory predict terms of order R^2 and higher (see *e.g.* de Alwis[7]). It occurs in the case of cosmic strings where higher order corrections, dependent on the "rigidity" of the string, contain higher derivatives[8,9], and in Dirac's relativistic model of the classical radiating electron.[2] Unlike lower derivative corrections, however, it is false to assume that adding a higher derivative correction term with a small coefficient will only perturb the original theory. The presence of an unconstrained higher derivative term, no matter how small it may naively appear, makes the new theory dramatically different from the original.

Unconstrained higher derivative theories have very distinctive features. As will be shown below, they have more degrees of freedom than lower derivative theories, and they lack a lower energy bound. There is nothing mathematically inconsistent with these features, but they make two almost identically looking theories, one a lower derivative theory, and the other the same theory with a higher derivative correction added, very different. The lack of a lowest energy state for the higher derivative theory is probably the most dramatic change. This *always* occurs when higher derivative terms are present (assuming no degeneracy or constraints), independently of how small their coefficients are. The addition of more degrees of freedom might be physically more accurate, but then it means that the original lower derivative theory was incomplete and missing (the most interesting) new families of solutions. It is particularly disturbing if there is a progression of higher order, higher derivative corrections, each system of which has more and more degrees of freedom. Classically, more degrees of freedom means that more initial data is required to specify motion. Quantum mechanically this means that for a particle x and $\dot{x}$ now commute since they are freely specifiable, and it becomes possible to measure the position and velocity at the same time. The momentum conjugate to x, π_x, still does not commute with x; $[x,\pi_x] = i\hbar$, but $\pi_x \neq m\dot{x}$. From the path integral point of view, the paths which dominate the functional integral are of a different class: where once they were nowhere differentiable, now they are everywhere once differentiable. Examples of all these behaviors are presented below. No familiarity with any of the properties of higher derivative theories is assumed.

There is a large class of theories naturally containing higher derivatives that do not suffer the above problems. Non-local theories, where the non-locality is regulated by a naturally small parameter, have perturbation expansions with higher derivatives. They avoid the above problems because they are constrained systems. They contain implicit constraints which keep the number of degrees of freedom constant and maintain a lower energy bound. Higher derivative theories that are truncated expansions of a non-local theory also avoid these problems, once the proper constraints are imposed. Any theory for which the higher derivative terms have been added as small corrections can be treated in the same manner, also avoiding the above problems.

Non-locality naturally appears in effective theories, valid only in a low energy limit and derived from a larger theory with some degrees of freedom frozen out. A good example is Wheeler-Feynman Electrodynamics,[10] in which the degrees of freedom of the electromagnetic field are frozen out. For two particles of mass m,

$$S = -\sum_i mc \int ds \left(\frac{dx_i^\mu}{ds}\frac{dx_{i\mu}}{ds}\right)^{1/2} + \frac{e_1 e_2}{c}\int ds\, ds' \frac{dx_1^\mu}{ds}\frac{dx_{2\mu}}{ds'}\delta(\Delta x^\nu \Delta x_\nu) \qquad (2-1)$$

where $\Delta x^\nu = x_1^\nu - x_2^\nu$. The only degrees of freedom remaining are of the charged particles. This is non-local because the particle-particle interaction is not instantaneous and point-like, but occurs in retarded time (action at a distance with finite propagation speed). The non-local Wheeler-Feynman theory is not valid for large $\frac{v}{c}$ (*e.g.* particle creation and annihilation is not allowed for), so there is a natural perturbative expansion in powers of $\frac{v}{c}$. Higher derivatives occur directly as a result of the non-locality. The action can be naturally expanded as[11]

$$S = \int dt \left[-\sum_i mc\left(1 - \frac{\mathbf{v}_i^2}{c^2}\right)^{1/2} - e_1 e_2 \sum_{p=0}^{\infty} \frac{(-D_1 D_2)^p}{2p!c^{2p}} \left(1 - \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{c^2}\right) r^{2p-1} \right] \quad , \tag{2 - 2}$$

where D_i signifies differentiation with respect to t of $\mathbf{x}_i$ only, and $r = |\mathbf{x}_1 - \mathbf{x}_2|$. To achieve the same solutions as the original Wheeler-Feynman theory, however, particular constraints must be imposed. Without the constraints, the expansion would have all the problems associated with higher derivative theories, which are not present in the Wheeler-Feynman theory. The effect of the constraints is to throw away "runaway" solutions. This is accomplished by only allowing solutions that can be Taylor expanded in powers of c^{-1} about $c^{-1} = 0$ (corresponding to infinite propagation speed).

These constraints allow the series expansion to be considered as a legitimate perturbative expansion. Without the constraints, higher order does not correspond to higher powers of $\frac{v}{c}$, but instead all terms contribute equally. With the constraints imposed, each term in the series contributes commensurately less as its order increases. For this reason the constraints will be referred to as "perturbative constraints".

The need for perturbative constraints was first pointed out Bhabha[12] in the context of Dirac's classical theory of the radiating electron[2] (and its higher order generalizations), although Dirac realized that runaway solutions should be excluded. The use of perturbative constraints as a method to remove the problems of higher derivative theories in general was discovered independently by Jaén, Llosa, and Molina (JLM)[13] and Eliezer and Woodard (EW)[14]. An explicit method for finding the perturbative constraints for any system expanded in a higher derivative series about some small expansion parameter was found also by JLM.[13] Given the perturbative constraints, a method of implementing them in a canonical fashion which greatly simplifies the calculation was found by EW.[14] Perturbative constraints can be implemented for either infinite or finite series expansions, though for the infinite case the perturbative constraints are already implicitly present if it is demanded that the equations of motion converge. For finite series expansions, where convergence of the series is not an issue, the perturbative constraints play an extremely important role. The finite series expansion, with perturbative constraints imposed, describes a system with the same solutions as those of the full non-local series (up to the appropriate order). The finite series expansion without the perturbative constraints describes a system with solutions most of which are nothing like the solutions to the original non-local system.

Finding the perturbative constraints does not depend on knowledge of the full non-local theory. It can be done just as easily if only a finite number of perturbative terms are known. For this reason it can equally be applied to any higher order system (with a small expansion coefficient) without knowing whether or not the theory is part of an infinite expansion. This is where the application of perturbative constraints is most powerful and most under-utilized.

In general relativity typical corrections take the form of curvature squared terms in the Lagrangian.[15-18] Even for small coefficients these terms can easily dominate the evolution of the system (as in Starobinsky inflation[19]). Applying the appropriate perturbative constraints describes a (different) system, in which the number of degrees of freedom are the same as in Einstein gravity, and which has no runaway solutions or ghostlike particles. It is a perturbative correction to Einstein gravity, which we know to be a very good approximation of nature.

Applying the perturbative constraints is not just an ad hoc procedure. It is completely natural and necessary in cases where the higher derivative theory is a truncated perturbative expansion of some larger, non-local (but otherwise well behaved) theory. The non-local theory itself may be the low energy effective limit of some even larger theory for which fields have been integrated out. It will be shown that the case of cosmic strings with higher derivative "rigidity" terms falls in this category.

There is sometimes a small cost to the use of perturbative constraints with higher derivative theories. Even for finite series expansions, locality can be lost under the influence of explicitly time-dependent sources. This is well know in the case of the self-interacting electron, where the non-local phenomenon of pre-acceleration (acceleration in response to a force that has yet to be applied) occurs. An example is demonstrated below. In the case of of the electron, this is considered unimportant, since it takes place only on the scale of the time light travels across the classical electron radius. At any rate, acausality only arises at scales for which the approximation of the electron as a classical particle breaks down. If a similar effect were to occur in a theory of gravity the non-locality would be at the Planck scale. Most physicists, though, would agree that at the Planck scale the usual notions of geometry probably break down (*e.g.* the appearance of space time foam), and so the possible presence of non-locality (and the accompanying loss of causality) should not be worrisome.

The structure of the article is as follows. First is a review of the behavior of unconstrained higher derivative theories in general, both classical and quantum, with simple examples of all interesting properties. Next is a discussion of various higher derivative theories that have been studied in the literature, including Dirac's classical electrodynamics, corrections to general relativity, and corrections to cosmic strings. This is followed by a discussion of the higher derivative theories that do not suffer from the above problems, and how the problems are avoided by the use of perturbative constraints.

Non-local systems, when cast into their higher derivative expansion, demand the use of perturbative constraints to reproduce the results of the original equations of motion. For any finite expansion approximation to the non-local theory, perturbative constraints are still required, and they must be imposed explicitly. The same finite expansion without the perturbative constraints would be a very different theory, completely unrelated to the original non-local theory in terms of its available solutions. The constraints must also be applied to systems where the purpose of the higher derivative terms is to provide small corrections to the original theory. Any theory which is intended (by construction or by physical motivation) to provide perturbative corrections to known solutions, but does not do so, is either incorrect or is being applied beyond its domain of applicability. The method of perturbative constraints is the only means by which a theory with higher derivative corrections can self-consistently avoid these problems.

Finally, the specific effects of applying the perturbative constraints are calculated for the cases of higher derivative extensions to general relativity and cosmic strings.

II. A Review of Higher Derivative Theories

(Many of the ideas in this review section are also covered in EW in a particularly lucid presentation.[14] All the equations presented here apply to one particle, one dimensional systems, but the generalization is trivial.) The Lagrangian formalism is straightforwardly applied to higher derivative theories. For a Lagrangian

$$L = L\left(q, \dot{q}, \ldots, q^{(N)}\right), \tag{2 - 3}$$

applying the variational principle gives

$$\delta S = \int_{t_i}^{t_f} dt \delta q \left(-\sum_{i=0}^{N} \left(-\frac{d}{dt} \right)^i \frac{\partial L}{\partial q^{(i)}} \right) + \sum_{i=0}^{N-1} p_{q^{(i)}} \delta q^{(i)} \Bigg|_{t_i}^{t_f} \quad , \tag{2 - 4}$$

where the $p_{q^{(i)}}$ are given by

$$p_{q^{(i)}} \equiv \sum_{k=i+1}^{N} \left(-\frac{d}{dt} \right)^{k-i-1} \frac{\partial L}{\partial q^{(k)}} \quad . \tag{2 - 5}$$

Assuming that the $\delta q^{(i)}$ are all held fixed at the boundary, the Euler-Lagrange equation is:

$$0 = \sum_{i=0}^{N} \left(-\frac{d}{dt} \right)^i \frac{\partial L}{\partial q^{(i)}} \quad . \tag{2 - 6}$$

The canonical formalism for higher derivative theories was developed by Ostrogradski[20]. The canonical momenta are defined by (2 - 5), which shows the generality of the Hamilton-Jacobi formalism. The Hamiltonian, as expected, is given by

$$\begin{aligned} H &= \sum_{n=0}^{N-1} p_{q^{(n)}} \dot{q}^{(n)} - L \\ &= \sum_{n=0}^{N-1} p_{q^{(n)}} q^{(n+1)} - L \quad . \end{aligned} \tag{2 - 7}$$

It is conserved and generates evolution in time, and so is equal to the energy of the system. Note that $q^{(N)} = q^{(N)}(q, \dot{q}, \ldots, q^{(N-1)}, p_{q^{(N-1)}})$ (assuming no degeneracy), but all the remaining $q^{(n)}$ are independent generalized coordinates and so are not inverted. For this reason, $L = L(q, \dot{q}, \ldots, q^{(N-1)}, p_{q^{(N-1)}})$ as well. The first order equations of motion are

$$\frac{\partial H}{\partial p_{q^{(n)}}} = \dot{q}^{(n)} = q^{(n+1)} \qquad \frac{\partial H}{\partial q^{(n)}} = -\dot{p}_{q^{(n)}} \qquad n = 0,1,\ldots,N-1 \,, \tag{2 - 8}$$

which reproduce the Euler-Lagrange equation. Note that it is entirely self-consistent from within this formalism to consider q and all its derivatives up to N completely independent. The dependence is regained from the equations of motion by the first relation of (2 - 8), which states

$$\frac{d}{dt} q^{(n)} = q^{(n+1)} \qquad \text{for} \quad n = 0,1,\ldots,N-2 \quad . \tag{2 - 9}$$

To demonstrate how higher derivative theories differ from their lower derivative counterparts, I will use the simple example:

$$L = \frac{1}{2}(1+\varepsilon^2\omega^2)\dot{x}^2 - \frac{1}{2}\omega^2 x^2 - \frac{1}{2}\varepsilon^2\ddot{x}^2 \tag{2 - 10}$$

which is a simple harmonic oscillator with the mass term slightly modified, and an acceleration squared piece. It may be helpful to think of $\varepsilon\omega \ll 1$, but this is never assumed in our calculations. The kinetic term has been modified only to make the calculations easier; it has no qualitative effect whatsoever, and all quantitative effects are small, $O(\varepsilon^2\omega^2)$. This contrasts strongly with the effects of the last term. It is tempting at first to view the last term as a small correction, but we shall see that this is false, independently of how small ε is. (The analogous example in scalar field theory has been examined by Hawking.[21])

That the number of degrees of freedom of a higher derivative theory is more than the lower derivative theory can be seen by examining equations (2 - 4) and (2 - 8). For the unconstrained system, there are $2N$ constants that determine the motion, corresponding to the $2N$ initial and final $q^{(n)}$s, or to the N initial (or final) $q^{(n)}$s and the N initial (or final) $p_{q^{(n)}}$s. This is a major qualitative difference from the lower derivative theory, which needs only two constants to specify the motion. This is also reflected in the quantum theory. The wave function has N arguments, and the commutation relations reflect the Ostrogradski canonical structure.

$$\left[q^{(n)}, p_{q^{(m)}}\right] = i\hbar\delta_{nm} \qquad \left[q^{(n)}, q^{(m)}\right] = 0 = \left[p_{q^{(n)}}, p_{q^{(m)}}\right]. \tag{2 - 11}$$

The second of these equations looks especially odd: the position and velocity of a particle commute! The wave function of the system will typically be functions of all the $q^{(n)}$, although one may, of course, Fourier transform any of the generalized coordinates and obtain it in terms of any of the conjugate momenta in their place.

Next we examine how these properties are exhibited in the example. The equation of motion is

$$\varepsilon^2 D^4 x + (1+\varepsilon^2\omega^2)D^2 x + \omega^2 x = 0 \qquad \text{where } D = \frac{d}{dt} \quad . \tag{2 - 12}$$

Being fourth order in time, it requires twice as many initial conditions as the $\varepsilon = 0$ case, independent of the size of ε. This is also reflected in the Ostrogradski canonical formalism, where the independent generalized coordinates are x and $\dot{x}$, and their respective generalized momenta are:

$$\pi_x = \frac{\partial L}{\partial \dot{x}} - D\left(\frac{\partial L}{\partial \ddot{x}}\right) = (1+\varepsilon^2\omega^2)\dot{x} + \varepsilon^2\dddot{x}$$
$$\pi_{\dot{x}} = \frac{\partial L}{\partial \ddot{x}} = -\varepsilon^2\ddot{x} \qquad . \qquad (2-13)$$

The Hamiltonian is

$$H = \frac{1}{2}\left(2\pi_x\dot{x} - \varepsilon^{-2}\pi_{\dot{x}}^{\,2} - (1+\varepsilon^2\omega^2)\dot{x}^2 + \omega^2 x^2\right) \qquad . \qquad (2-14)$$

Note the impossibility of taking the $\varepsilon \to 0$ limit in this case. The general solution is

$$x = A_+\cos(\omega t + \phi_+) + A_-\cos(\varepsilon^{-1}t + \phi_-) \qquad . \qquad (2-15)$$

For $\varepsilon\omega \ll 1$, the second mode oscillates extremely rapidly. The modes separate exactly because the Lagrangian is quadratic in all terms; non quadratic terms would couple the modes. The oscillatory nature of the second term is not related to the fact that the $\varepsilon = 0$ case is a simple harmonic oscillator: in the case $\omega = 0$, the solution is $x = x_0 + v_0 t + A_-\cos(\varepsilon^{-1}t + \phi_-)$. In the case $\varepsilon^2 < 0$, the solution is $x = A_+\cos(\omega t + \phi_+) + A_c\cosh(|\varepsilon|^{-1}t) + A_s\sinh(|\varepsilon|^{-1}t)$.

Quantum mechanically, since x and $\dot{x}$ are independent coordinates, the wave function will be a function of both: $\psi = \psi(x,\dot{x})$ (though we could also use $\psi = \psi(x,\pi_{\dot{x}}), \psi = \psi(\pi_x,\dot{x})$, or $\psi = \psi(\pi_x,\pi_{\dot{x}})$). Note that $[x,\dot{x}] = 0$, allowing the position and velocity to measured in the same experiment to arbitrary accuracy. This is also independent of the size of ε, so long as it is non-vanishing.

The quantum mechanical system is solved exactly in Appendix A. The energy eigenstates are labeled by two non-negative integers.

$$E = (n + \tfrac{1}{2})\omega - (m + \tfrac{1}{2})\varepsilon^{-1} \qquad \text{for } n,m = 0,1,2,\ldots \quad . \tag{2 - 16}$$

The simplest wave function to calculate is:

$$\psi_{00}(x,\dot{x}) = \left(\frac{\omega}{\varepsilon\pi^2}\right)^{1/4} \times \exp\left(-\frac{\omega^2(1-\varepsilon^2\omega^2)x^2 - 4i\varepsilon\omega^2 x\dot{x} - (1-\varepsilon^2\omega^2)\dot{x}^2}{2\varepsilon^{-1}(1+\varepsilon^2\omega^2)}\right) \tag{2 - 17}$$

As expected, the limit $\varepsilon \to 0$ does not approach the purely simple harmonic oscillator ground state wave function.

Strongly related to the fact that $[x,\dot{x}] = 0$ is that the class of paths that dominate the Feynman path integral changes. The path integral sums over all possible paths, but a particular class of paths dominate the sum, which can be seen by examining expectation values in transition amplitudes.[22] First we examine the properties of these paths for a lower derivative theory. For a path integral skeletonized into time slices of duration δ, the expectation value of the distance crossed in that time is approximately

$$\langle \Delta x \rangle \sim \delta^{1/2} \tag{2 - 18}$$

So, as $\delta \to 0$, the typical paths (averaged with a complex weighting) are continuous. But the expectation value of the particle's velocity diverges

$$\langle \dot{x}_{cl} \rangle \sim \left\langle \frac{\Delta x}{\delta} \right\rangle \sim \delta^{-1/2} \quad . \tag{2 - 19}$$

For a higher derivative theory, this is not true. The typical paths for acceleration dependent Lagrangians have finite velocities, but their acceleration diverges, i.e. the paths are continuous in $(x, \dot{x})$ space. The higher the derivatives in the Lagrangian, the smoother the paths become. An infinite number of higher derivatives would have, in some sense, only perfectly smooth paths contributing. (In fact, because the path integral formulation can be used to derive Schroedinger's equation, one can read off expectation values from the Hamiltonian, as done by Feynman[22], and as shown in Appendix B.)

To illustrate the path integral properties of higher derivative Lagrangian, we will use the simpler case $\omega = 0$: a free particle with a (seemingly small) quadratic acceleration term. (The $\omega \neq 0$ case is conceptually no more difficult but requires enormously more calculation.)

$$\begin{aligned} L &= \frac{1}{2}(\dot{x}^2 - \varepsilon^2 \ddot{x}^2) \\ H &= \frac{1}{2}(-\varepsilon^{-2}\pi_{\dot{x}}{}^2 + 2\pi_x \dot{x} - \dot{x}^2) \end{aligned} \tag{2 - 20}$$

We calculate the following transition expectations

$$
\begin{aligned}
&\langle \Delta x \rangle \sim \dot{x}\delta \to 0 \\
&\langle \Delta \dot{x} \rangle \sim \varepsilon^{-1}\delta^{1/2} \to 0 \\
&\langle x_{cl} \rangle \sim x \\
&\langle \dot{x}_{cl} \rangle \sim \dot{x} \;\&\; \frac{\langle \Delta x \rangle}{\delta} \sim \dot{x} \\
&\langle \ddot{x}_{cl} \rangle \sim \frac{\dot{x}}{\delta} \;\&\; \frac{\langle \Delta \dot{x} \rangle}{\delta} \;\&\; \frac{\langle \Delta x \rangle}{\delta^2} \to \infty \quad .
\end{aligned}
\tag{2 - 21}
$$

The dominant paths are now once differentiable. The exact propagator has also been calculated for this system using modes, as shown in Appendix C.

Another extremely important property of higher derivative theories, both classical and quantum, is the lack of any lower energy bound. This can be seen most easily through (2 - 7). The only dependence on the $p_{q^{(n)}}$ for $n < N-1$ is linear, permitting the Hamiltonian to take on arbitrarily negative values. This carries over into the quantized system as well.[14] This property is easily demonstrated by our example system: the Hamiltonian is manifestly indefinite in equation (A - 2). The energy for the general solution given in equation (2 - 15) is

$$
E = \frac{1}{2}(1-\varepsilon^2\omega^2)(\omega^2 A_+{}^2 - \varepsilon^{-2} A_-{}^2) \quad , \tag{2 - 22}
$$

which is also manifestly indefinite. The effect of even a small amplitude for the negative mode leads to enormously negative energies (for $\varepsilon\omega \ll 1$). Even though exciting the negative energy modes leads only to oscillatory behavior (for the $\varepsilon^2 > 0$ case), it is nevertheless unstable since even small excitations of those modes lowers the energy dramatically. Any coupling present in a not purely quadratic

Lagrangian system would make the problem worse. The quantized system has the same negative energy problems, as seen in (2 - 16) and Appendix A. Attempts have been made within quantum mechanics to change the minus sign in (2 - 16) into a plus by giving half of the quantum states negative norm.[23,21] This merely shifts the problem from from the lack of a ground state to the lack of unitarity (arising from the now possible zero norm modes), but it is really the same problem transformed.

Higher derivative field theories have the related problem of ghosts: excitations (particles) of negative energy (mass) (see *e.g.* Hawking[21]). They behave analogously to the oscillatory excitations of negative energy states in our example. Creation of ghost particles not only costs no energy, it produces excess energy, causing them to be spontaneously produced in infinite numbers.

In short, the distinct features of higher derivative theories fall into two major categories, either deriving from the more numerous degrees of freedom than the lower derivative case, or from the loss of a lowest energy state. It should be noted that there is nothing fundamentally contradictory or mathematically inconsistent with higher derivative systems. A good example of this kind of theory is the pure R^2 theory of Horowitz[24] (although there are still problems with the negative energy modes, as pointed out by Eliezer and Woodard[14]).

These features do become serious problems in most cases, however. Except in a purely cosmological context, the lack of a ground state is very unphysical. It is also unphysical when there is a sequence of higher order theories for which the higher order terms are supposed to provide small corrections, but instead introduce new degrees of freedom and new behavior at every step. In our example above, the problems become manifest when the system is compared to a simple harmonic oscillator ($\varepsilon = 0$). These problems cannot be avoided in unconstrained higher derivative theories, whether oscillating particles, flexing cosmic strings, or R^2 gravity.

III. Naturally Occurring Higher Derivative Theories

As stated in the introduction, higher derivative theories appear naturally in at least two contexts. The first is as corrections to a lower derivative theory. The oldest example of this is the Abraham-Lorentz model of a non-relativistic, classical, radiating, charged particle (see *e.g.* Jackson[1]) and the relativistic generalization due to Dirac.[2] In attempting to take into account the loss of energy due to radiation, a third derivative term is introduced into the equation of motion. The higher derivative term has a small coefficient, $\tau = \frac{2}{3}\frac{e^2}{mc^3} \sim 10^{-23}\,\text{sec}$, yet there are now twice as many solutions as for the non-radiating electron, and half the solutions are runaways: solutions qualitatively different from solutions of the non-radiating electron. (This is a dissipative system due to the radiation, so the lack of a lower energy bound is not manifest.) As an example, Dirac's equation of motion in the absence of external forces

$$\dot{v}_\mu = \tau(\ddot{v}_\mu - \dot{v}_\nu \dot{v}^\nu v_\mu) \qquad \text{where } \eta_{\mu\nu} = (-,+,+,+) \tag{2 - 23}$$

has the general solution (for motion in the x direction)

$$\begin{aligned} v_x &= \sinh(e^{s/\tau} + b) \\ v_t &= \cosh(e^{s/\tau} + b) \end{aligned} \tag{2 - 24}$$

where s is proper time and b is an integration constant, or

$$v_\mu = \text{constant} \ . \tag{2 - 25}$$

For the first solution, the free electron accelerates to near the speed of light in a time comparable to τ. For the second, the electron remains unaccelerated, which is the expected answer for zero external force.

Another example of a naturally occurring higher derivative theory is the case of cosmic strings. If treated as an unconstrained higher derivative theory, as is always done in the literature, it suffers from all the above problems. The number of degrees of freedom is dependent on which order the higher order expansion is stopped. The excitations of the newly available modes contain negative energy, just as in the case of the oscillator above. Note that this is *independent of the sign* of the coefficient of the higher derivative term (corresponding to rigidity). It is completely analogous to the acceleration dependent harmonic oscillator example above. For $\varepsilon^2 < 0$ the negative energy modes are exponential in time and obviously unstable, but even for $\varepsilon^2 > 0$, exciting the negative energy modes allows arbitrary amounts of energy to be extracted. Even a small kick ($A_- \ll A_+$ in the example) can extract large amounts of energy since the negative energies are inversely proportional to the small parameter, in this case the width of the string. We shall see below that the perturbative constraints must be applied for consistency.

Higher order corrections to general relativity itself can arise either quantum mechanically or classically. Quantum mechanically, conformal anomalies give rise to an effective action with higher derivative terms (see *e.g.* Birrell and Davies[6]), which can be local ($\propto R^2$) or non-local (not expressible in local quantities such as the metric and Riemann tensor). Renormalizability arguments demand the presence of terms $\propto R^2$ and $\propto R_{ab}R^{ab}$.[17] The effect of these terms is to give additional families of solutions, some of which are "runaways". The negative energy problem manifests itself when coupled to matter; there is in general no positive energy theorem[16] (with the exception of special initial conditions for certain higher derivative terms[25]). There are also, in general, problems with ghost fields and local instabilities due to the presence of negative energy modes.[15] The concept of a runaway solution on a cosmological scale is somewhat unclear in the case of gravity where, with an ordinary cosmological constant, exponential inflation is a physical solution. Nevertheless, the smaller the coefficient of the higher derivative terms, the faster the rate of inflation it can induce. These extra solutions are non-Taylor expandable in powers of the small coefficient. Lovelock gravities[26,27] have similar problems, despite the fact that they are, strictly speaking, not higher derivative theories. Lovelock theories contain higher order terms in the Lagrangian which are dimensionally extended Euler densities. They allow extra solutions to the field equations, though the number of new solutions is finite, not a continuous family. Nevertheless, some of the new solutions are dramatically different from the original and can be considered runaways.

Classically, string theory gives higher order (local) corrections to the action in higher powers of curvature and its derivatives, as shown below in (2 - 50). If left unconstrained, in addition to all the problems of the preceding paragraph, each theory obtained by truncating the expansion at a given order has a different number of degrees of freedom than the previous one.

IV. Higher Derivative Theories Without The Problems

When higher derivative theories occur as a result of truncating a perturbative expansion of a non-local theory, the usual problems of higher derivative theories do not occur because the perturbative constraints must be applied. They guarantee that of all possible solutions to the unconstrained higher derivative equation of motion, only solutions that are Taylor expandable in the small expansion parameter are permitted. All other solutions are considered runaways, solutions that do not exist in the limit of zero expansion parameter. This corresponds to the limit of infinite propagation speed or instantaneous interactions in the Wheeler-Feynman model. The extra solutions have extremely rapid behavior for small expansion coefficients and are always associated with negative energy behavior. The remaining solutions form a two parameter family.[13] The first use of the exclusion of runaway solutions was in the context of removing obviously unphysical solutions such as (2 - 24) from the Dirac model (and its non-relativistic variations). It was suggested that runaway solutions be isolated and defined by their late time behavior, that the acceleration should be finite in the far future for finite forces. Imposing future boundary conditions at large scales is undesirable, since, if there is acausality present in the universe, it is likely to be only at the smallest (i.e. Planck) scales. Furthermore, the finite acceleration criterion does not generalize well to other classical theories, such as general relativity, which has extremely varied cosmological solutions. Bhabha pointed out that for Dirac's theory (and higher order extensions), all non-runaway solutions are Taylor expandable in the natural small expansion parameter of the theory τ. Imposing the perturbative constraints is equivalent to throwing away runaway solutions, but relieves us of the obligation to specify future conditions.[12]

Non-local systems, such as the Wheeler-Feynman theory, when cast into their higher derivative expansion (first done for the Wheeler-Feynman theory by Kerner[11]), demand the use of perturbative constraints to reproduce the results of the original equations of motion. The perturbative constraints are implicit in the Lagrangian by demanding convergence. This will be demonstrated below with a simple model. For any finite expansion approximation to the non-local theory, perturbative constraints are still required, but must be imposed explicitly, since they are no longer demanded by convergence of the series. The same finite expansion without the perturbative constraints is a very different theory, completely unrelated to the original non-local theory in terms of its available solutions.

It is also perfectly self-consistent and valid to impose perturbative constraints on a higher derivative system even without the sure knowledge that it is a truncated expansion of some non-local theory. This was done by Dirac when he threw away the undesirable runaway solutions. It is not unreasonable to apply it to any higher derivative theory for which the method is applicable and examine the consequences. It is absolutely necessary if the higher derivative theory is to at all resemble the original lower order theory in its behavior.

The JLM procedure for finding the perturbative constraints strictly imposes the condition that all solutions must be Taylor expandable in the perturbative expansion parameter.[13] It is not possible to invert all of the canonical momenta within the limits of Taylor expandability, which signals the presence of a primary constraint. Secondary constraints are obtained by taking time derivatives of the primary constraints. Linear combinations of the constraints can always be put in the form

$$x^{(i)} - f_i(x,\dot{x}) = 0 \qquad i = 2,\ldots,N \quad . \qquad (2\text{-}26)$$

Add the constraints in this form to the Lagrangian with Lagrange multipliers, and proceed either to the Euler-Lagrange equations or the Hamilton-Dirac equations (whose equivalence for constrained higher derivative systems has been shown by Pons[28]).

There is no issue of whether or not to quantize on a larger phase space and apply weak constraints afterwards, because there is no larger phase space. The perturbative constraints are second class constraints, and hold strongly. One must use the Dirac procedure, with Dirac brackets instead of Poisson brackets, calculate the Hamiltonian, and use the Hamilton-Dirac equations to quantize the system.[29,30]

There is an easier method to put the system in canonical form, however, based on the fact that we know what form the final answer must take, due to EW[14] and quickly reviewed here. There exists a local, lower order theory equivalent to the higher derivative theory plus the constraints.[30] By knowing the energy, reduced to a function of x and $\dot{x}$ only, $E_r(x,\dot{x})$ (once the constraints have been applied), and knowing that the Hamiltonian is equal in value to the energy, the value of p, the canonical momentum to x, for a reduced version of the same system can be inferred. In order that

$$\dot{x} = \{x, H_r(x,p)\} \tag{2 - 27}$$

hold true, we must have

$$p(x,\dot{x}) = \int_0^{\dot{x}} \frac{dv}{v} \frac{\partial E_r(x,v)}{\partial \dot{x}} + p(x,0) \quad . \qquad (2 - 28)$$

(There may be some uncertainty in the choice of $p(x,0)$, but this corresponds to the uncertainty in the initial Lagrangian of whether to add total derivatives of the form $\Delta L = \frac{dF(x(t))}{dt}$, which one always has the freedom to do. This addition corresponds to the making the canonical transformation

$$x' = x \qquad p' = p + \frac{\partial F}{\partial x} \quad .) \qquad (2 - 29)$$

We can then invert (2 - 28) to get $\dot{x}(x,p)$ and arrive at at $H_r(x,p) = E_r(x,\dot{x}(x,p))$ and $L_r(x,\dot{x}) = p(x,\dot{x})\dot{x} - E_r(x,\dot{x})$. Using the new x and p, where $\{x,p\} = 1$, is equivalent in every way to using all the $x^{(n)}, p_{x^{(n)}}$,and constraints, with Dirac brackets[30].

In quantum cosmology there is not, in general, a special physical quantity that takes the role of time which is so essential to Hamiltonian based quantum mechanics. In these cases the action is taken to be the fundamental basis and quantization can proceed using the Feynman sum over histories approach. The technique is to Euclideanize the action and sum over all paths with a specified boundary condition, which produces a specific quantum state. This does not require any canonical formalism. If there is a special time parameter or symmetry that allows a useful canonical formalism, the EW method can be used for ease of calculation of the action, but is not required.

Let us observe all the above properties and characteristics of a non-local theory with a simple model. The model is solved exactly, both classically and quantum mechanically, and can be expanded in an infinite or arbitrarily truncated series in higher derivatives. The system is a one dimensional harmonic oscillator but for which the potential depends not only on the position at a given time, but at all times past and future. Distant times, however, contribute exponentially weakly. We begin with the equation of motion

$$\begin{aligned} 0 &= \ddot{x} + \omega^2 \int_0^{\infty} ds\, e^{-|s|} \frac{1}{2}[x(t+\varepsilon s) + x(t-\varepsilon s)] \\ &= \ddot{x} + \omega^2 \int_{-\infty}^{+\infty} \frac{ds}{2} e^{-|s|}\, x(t+\varepsilon s) \end{aligned} \qquad (2 - 30)$$

When $\varepsilon\omega \ll 1$ we might expect this system's behavior to be very similar to a simple harmonic oscillator, and indeed this is the case. Using the ansatz $x = A e^{i\gamma t}$ we find two roots

$$\gamma^2 = \begin{cases} \omega^2 \chi^2 \\ -\varepsilon^{-2} \chi^{-2} \end{cases} \text{ where } \chi = \left(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{1 + 4\varepsilon^2\omega^2} \right)^{-\frac{1}{2}} = 1 - \tfrac{1}{2}\varepsilon^2\omega^2 + \dots \quad . \qquad (2 - 31)$$

The second root, when reinserted into the equation of motion, fails to converge, and so is not a solution. The remaining root corresponds to the solution we expect: harmonic motion with frequency close to the original, $\gamma = \omega(1 + O(\varepsilon^2\omega^2))$. The general real solution is

$$x = A\cos(\gamma t + \phi) \quad . \qquad (2 - 32)$$

To put the system in to a Lagrangian form, we expand out the equation of motion into an infinite series

$$0 = \ddot{x} + \omega^2 \sum_{n=0}^{\infty} (\varepsilon D)^{2n} x \tag{2 - 33}$$

and we can construct a Lagrangian that will give us this.

$$
\begin{aligned}
0 &= -\sum_{n=0}^{\infty} (-D)^n \frac{\partial L}{\partial (D^n x)} \\
L &= \frac{1}{2}\left\{ \dot{x}^2 - \omega^2 x^2 + \omega^2 \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \left[(\varepsilon D)^{2n+1} x (\varepsilon D)^{2m+1} x \right.\right. \\
&\qquad \left.\left. + (\varepsilon D)^{2n+2} x (\varepsilon D)^{2m+2} x \right] \right\} \qquad (2 - 34) \\
&= \frac{1}{2}\left\{ \dot{x}^2 - \omega^2 x^2 + \omega^2 \left[\left(x - \int_{-\infty}^{+\infty} \frac{ds}{2} \mathrm{e}^{-|s|} x(t+\varepsilon s) \right)^2 \right.\right. \\
&\qquad \left.\left. + \left(\int_{-\infty}^{+\infty} \frac{ds}{2} \mathrm{e}^{-|s|} \varepsilon \dot{x}(t+\varepsilon s) \right)^2 \right] \right\}
\end{aligned}
$$

There are of course other Lagrangians that give us the same equations of motion. For instance we can always add the total derivative $\frac{d}{dt} f(x)$ without changing the classical equation of motion. When adding total derivatives with higher derivatives, however, one must exercise caution, or the variational principle necessary for quantization will be lost. Details of the need for a valid variational formulation are discussed in Appendix D.

Since there is only one true degree of freedom (i.e. the evolution is specified completely by x_i and $\dot{x}_i$), not infinitely many as implied by the infinite expansion, there must be an infinite number of constraints implicit in the expansion Lagrangian. They must express all higher derivatives in terms of x and $\dot{x}$, and can be put in the form of (2 - 26). By inspection of the known solution (2 - 32) they must be

$$\left.\begin{aligned} D^{2n+2}x &= (-1)^{n+1}\gamma^{2n+2}x \\ D^{2n+3}x &= (-1)^{n+1}\gamma^{2n+2}\dot{x} \end{aligned}\right\} \quad n = 0,1,\ldots \quad , \tag{2 - 35}$$

or some linear combination. The JLM procedure is unnecessary here because we have the general solution (2 - 32), and finding constraints to enforce it can be done by inspection. These constraints may be put explicitly into the Lagrangian with Lagrange multipliers. It is not necessary, since the convergence of the series enforces the constraints implicitly, but it is helpful to acknowledge them explicitly as well.

The energy is calculated using the Ostrogradski Hamiltonian.

$$\begin{aligned} E &= \sum_{n=1}^{\infty}\sum_{m=0}^{n-1}(D^{n-m}x)\left[(-D)^m \frac{\partial L}{\partial(D^n x)}\right] - L \\ &= \frac{1}{2}\dot{x}^2 + \omega^2 x\int_{-\infty}^{\infty}\frac{ds}{2}\mathrm{e}^{-|s|}\,x(t+\varepsilon s) \\ &\qquad\qquad -\frac{1}{2}\omega^2\int_0^{\infty}ds\,\mathrm{e}^{-s}\,x(t+\varepsilon s)\int_0^{\infty}ds'\,\mathrm{e}^{-s}\,x(t-\varepsilon s') \end{aligned} \tag{2 - 36}$$

Note that $\lim_{\varepsilon\to 0} E = \frac{1}{2}(\dot{x}^2 + \omega^2 x^2)$, as expected.

The next step is to quantize the system. As usual, finding the ground state of the system (which would not exist if it really were an unconstrained higher derivative theory) can be done without reference to canonical formalism or Poisson/Dirac brackets. The ground state can be calculated using the Euclidean sum over paths.

$$\psi_0(x_f) = \int \mathscr{D}\tilde{x}\, \mathrm{e}^{-I[\tilde{x}(\tau)]} \tag{2 - 37}$$

where the Euclidean action $I = \iota S$, $t = -\iota\tau$, the sum is over all paths of finite Euclidean action ending at x_f, and $\tilde{x}(\tau) = x(t)$. We may take the final Euclidean time to be $\tau_f = 0$, without loss of generality.

$$I = \frac{1}{2}\int_{-\infty}^{0} d\tau \left\{ \dot{\tilde{x}}^2 + \omega^2\tilde{x}^2 - \left[\left(\tilde{x} - \int_{-\infty}^{+\infty} \frac{ds}{2} \mathrm{e}^{-|s|}\, \tilde{x}(t + \iota\varepsilon s) \right)^2 \right.\right.$$
$$\left.\left. - \left(\int_{-\infty}^{+\infty} \frac{ds}{2} \mathrm{e}^{-|s|}\, \varepsilon\dot{\tilde{x}}(t + \iota\varepsilon s) \right)^2 \right] \right\} \tag{2 - 38}$$

The action is still non-local in real time, even though the paths are in Euclidean time. The path integral can be done exactly since the action is quadratic in x.[22] There is only one classical solution with finite Euclidean action, $x_{cl} = x_0\,\mathrm{e}^{\gamma\tau}$. Let $\tilde{x} = x_{cl} + q$. Then

$$\begin{aligned} \psi_0(x_0) &= \underbrace{\int \mathscr{D}q\, \mathrm{e}^{-I[q]}}_{\text{constant}}\, \mathrm{e}^{-I[x_{cl}]} \\ &= (\text{const})\,\mathrm{e}^{-\frac{1}{2}\gamma\eta x_0^2} \end{aligned} \tag{2 - 39}$$

where $\eta = 2 - \chi^2 = 1 + O(\varepsilon^2\omega^2)$, so $\psi_0(x) \propto \mathrm{e}^{-\frac{1}{2}\omega x^2(1+O(\varepsilon^2\omega^2))}$ as expected.

This system can be put into canonical form in two ways: expanding the non-local integrals into infinite sums, defining the momenta by equation (2 - 5), applying the (second class) constraints of equation (2 - 35) with Lagrange multipliers, and using Dirac brackets, or by the calculationally much simpler EW method discussed above. The reduced relevant quantities given by this method are

$$\begin{aligned} E_r &= \tfrac{1}{2}\eta(\dot{x}^2 + \gamma^2 x^2) \\ p &= \eta\dot{x} \\ H_r &= \tfrac{1}{2}(\eta^{-1}p^2 + \eta\gamma^2 x^2) \\ L_r &= \tfrac{1}{2}\eta(\dot{x}^2 - \gamma^2 x^2) \quad . \end{aligned} \tag{2 - 40}$$

The function $p(x,0)$ is determined in this case by demanding $L(x_{classical}) = L_r(x_{classical})$, which gives $p(x,0) = 0$. Now we know the whole system is canonically equivalent to a simple harmonic oscillator, so in particular,

$$\begin{aligned} H\psi_n &= \tfrac{1}{2}\gamma(n+1)\psi_n \quad \text{where} \quad n = 0,1,\ldots \\ \psi_0 &= c\,e^{-\frac{1}{2}\eta\gamma x^2} \quad , \end{aligned} \tag{2 - 41}$$

which agrees with the Euclideanized sum over histories calculation of the ground state wave function.

Now suppose we are not given the full theory, but only the first N terms. We may not even know where they came from. But we do know that the zeroth order term is a good approximation when $\varepsilon\omega \ll 1$. Or perhaps we do not have the tools to solve the full theory, but only for the first N terms.

$$L_N = \tfrac{1}{2}\{\dot{x}^2 - \omega^2 x \sum_{n=0}^{N} (\varepsilon D)^{2N} x\} + \text{ total derivatives} \quad . \tag{2 - 42}$$

The cases $N = 0,1$ are trivial, and the solutions are the same as the solution to the full non-local theory, to order $(\varepsilon\omega)$ and $(\varepsilon\omega)^2$ respectively. If left unconstrained, however, the case $N = 2$ has all the quirks and problems of the acceleration dependent oscillator above: twice as many solution as the zeroth order case, negatively unbounded energy, etc. For $N = 3,4,...$, the number of solutions continues to increase and all associated problems get worse.

The perturbative constraints are needed explicitly here (they are implicit in the full theory). The new "solutions" to the unconstrained finite series approximations do not converge when put in the equation of motion for the full theory. The appropriate constraints can be obtained by the JLM procedure, which is necessary when the full theory is not known, and gives

$$\ddot{x} + \gamma_N{}^2 x = 0 \qquad (2 - 43)$$

where γ_N depends on the expansion order N, and $\gamma_N{}^2 - \gamma^2 = O\left((\varepsilon\omega)^{2N+2}\right)$. Higher derivative constraints are obtained by differentiating and substituting as necessary. Solving this system, for finite N, gives the correct solutions of the full infinite system (to the appropriate order), whether classical or quantum, whether via the Lagrangian or Hamiltonian. Solving the system without the perturbative constraints, while describing a well-defined system, does not approximate the full non-local system in any sense.

The above non-local oscillator is an example of a perfectly well behaved system that appears sick when expanded naively in a perturbation series. But when expanded properly, with the knowledge that the only contributing solutions are those close to the zeroth order solutions, the expansion is useful, perturbative, physical, and agrees with the full theory to the appropriate order.

The one important aspect of higher derivative and non-local systems that does not appear in the above example is the appearance of acausal solutions, i.e. pre-acceleration types of effects. These appear when the system is coupled to explicitly time dependent terms. The best known example is Dirac's classical electron. In the case of non-zero force, the electron experiences a pre-acceleration on times of the order of τ.[1,2] For a one dimensional delta function impulse the equation of motion (2 - 23) in the non-relativistic limit becomes

$$\tau\dddot{x} - \ddot{x} = \kappa\delta(t) \tag{2 - 44}$$

which has the general solution $\dot{x} = c_1 e^{t/\tau} + c_2$ for $t \neq 0$ with an instantaneous change in $\ddot{x}$ of κ across $t = 0$. For $t > 0$, $c_1 = 0$ by requiring finite acceleration in the infinite future. For $t < 0$, $c_2 = 0$ if we desire zero velocity in the far past. The solution is

$$\dot{x} = \begin{cases} \kappa\tau e^{t/\tau} & t < 0 \\ \kappa\tau & t > 0 \end{cases} \tag{2 - 45}$$

which has non-zero acceleration before the force is applied, but the time scale it occurs on is τ. In general, acausality only appears at the scale of the small expansion parameter, and as shown by Wheeler and Feynman for similar theories, the acausality decreases as the number of particles rises.[31] At any rate, for theories in which the non-locality appears only as a low energy effective theory, the theory itself is only an approximation at that scale, and its results at that scale will reflect this. If the theory is truly non-local, as the true Theory Of Everything might be (we have no experimental evidence whether or not nature is local near the Planck scale), then the non-locality will be manifest in the solutions.

The JLM procedure fails in the presence of external sources, though a related procedure has been proposed.[32] Note that (2 - 45) is not analytic in in τ, and so whatever method is needed to remove ill-behaved solutions will most likely not allow a solution of this form. The acausality may still manifest itself, though probably in a different way. The form of the source term might be restricted due to (non-local) back reaction, on the order of the expansion parameter.

It is important to mention and put to rest a common fallacy, that the extra degrees of freedom of higher derivative theories are somehow related to the degrees of freedom frozen out in creating the effective non-local theory. For instance, that the higher derivative degrees of freedom in a cosmic string arise from the lost degrees of freedom of the scalar and gauge fields, or that the higher derivative degrees of freedom from curvature squared corrections to general relativity arise from the frozen-out massive string modes. The field degrees of freedom frozen out to create the non-local effective theory are gone and cannot be regained. The apparent higher derivative degrees of freedom, as in the model above, are artifacts from trying to perturbatively expand a non-local theory without the necessary perturbative constraints. This is also particularly relevant in the case of cosmic strings, where negative energy modes are available in the unconstrained higher derivative case, yet exact traveling wave solutions along a string in the full field theory have only positive energy.[33]

V. Effects on General Relativity, Cosmic Strings, and Other Theories

Under what circumstances should perturbative constraints be applied? They must always be applied in the case where the theory in question is known to be a truncated expansion (with a small coupling constant) of a non-local theory. The non-local theory may itself be a low energy limit of some larger local theory. Specific cases of this are Wheeler-Feynman electrodynamics and cosmic strings (as will be shown below). If the expansion is to be perturbative, then the perturbative constraints are the only means of enforcing it. To verify whether the perturbative expansion itself is appropriate, check the behavior of the zeroth order approximation (*e.g.* a slowly moving electron for which radiation effects are ignored, or a very straight slow cosmic string). If it is appropriate to approximate the system with only the zeroth order term, then it is appropriate to use higher order perturbative corrections, and hence the perturbative constraints. This is certainly the case for Wheeler-Feynman electrodynamics and cosmic strings without too much curvature (i.e. no kinks or cusps). Where the zeroth order approximate theory is inappropriate (*e.g.* electrons with speeds near c and intersecting cosmic strings), the perturbative expansion is inappropriate as well. The expansion without perturbative constraints is never appropriate.

Consider the case of Cosmic strings. The usual derivation of the string action[9,8] begins with the full gauge theory

$$S = \int d^4x \sqrt{-g}\, \mathscr{L}(\phi, \partial_\mu \phi, A_\mu, F_{\mu\nu}) \quad . \tag{2 - 46}$$

Let $\phi = \phi_0(x^\mu)$, $A_\nu = A_{\nu 0}(x^\mu)$ be a field configuration that describes a string, and let $x^\mu = X^\mu$ be the location in spacetime of the string. We want to write down an effective action based only on the movement of the string, i.e. formed only from functions and operators acting on X^μ. First pick a coordinate system such that two coordinates ξ^a are in the world sheet of the string and two coordinates ρ^A are Gaussian normal coordinates perpendicular to the world sheet

$$x^\mu = X^\mu(\xi^a) + \rho^A n^\mu_A \tag{2 - 47}$$

where n^μ_A are two (arbitrary) unit vectors normal to the world sheet, $a = 1,2$, and $A = 1,2$. The action, still exact, now reads

$$S = \int d^2\xi\, d^2\rho [\mathscr{L}(\phi_0, A_{\mu 0}) + \ldots]\, [\sqrt{-\gamma}(1 + \rho^A K_A + \tfrac{1}{2!}\rho^A \rho^B K_A K_B + \ldots)] \tag{2 - 48}$$

where $\gamma_{ab}(\xi^a)$ is the metric on the world sheet and $K_A(\xi^a)$ are the traces of the two extrinsic curvatures. We can make an effective, non-local theory by integrating out all degrees of freedom off the string world sheet. It is non-local because the string has finite thickness, so the energy of a piece of the string propagates not only along the string but also over and around it, but those extra degrees of freedom are now frozen out of the picture. Once we have done this, the lost degrees of freedom cannot be recovered, i.e. we cannot reconstruct the original system from the effective theory. This effective, non-local theory is not usually examined *per se*, but is itself perturbatively expanded, effectively in powers of the string width multiplied by the extrinsic curvatures

$$S_{pert} = -\mu \int d^2\xi \sqrt{-\gamma}(1 + \tfrac{s_0}{\mu} K^A K_A + \ldots) \qquad (2-49)$$

where μ is the string tension, s_0 is the "rigidity", etc.. The zeroth order term is just the Nambu-Goto action. Higher order expansions contain higher derivatives via the extrinsic curvatures and their derivatives. If left unconstrained, these higher derivative terms would have the usual disastrous effect, making the so called perturbative theory totally different from the full non-local theory. Instead, enforcing the perturbative constraints produces solutions consistent with the full gauge field theory, allows the expansion to be truly perturbative, and removes all the problems of extra degrees of freedom and negative energy.

For cosmic strings the perturbative constraints ensure that all higher order solutions remain close to the solutions of the old, zeroth order , Nambu-Goto action. Since it has been shown that solutions of the Nambu-Goto action are also solutions when the first order "rigidity" corrections are present,[8] the appropriate constraint, for a non-interacting string, is that the original equations of motion remain unchanged. The result is that, for an isolated string, the rigidity term has no effect at all on the motion, *independently of the sign of its coefficient.* The first contributing corrections to the Nambu-Goto action must come from higher order terms, *e.g.* torsion or derivatives of extrinsic curvature. As in the case of Dirac's theory, however, the rigidity term will certainly play a role once external forces are considered.

Similar to the cases of non-radiating electrons and cosmic strings, Newtonian gravity and its post-Newtonian corrections can be derived from a perturbation expansion of an effective non-local theory ultimately derived from Einstein gravity. One might expect the same phenomenon to occur, since the effects of gravitons have been integrated out. Acceleration dependent terms do occur in the post-Newtonian and post-post-Newtonian approximations, but only linearly, which is a degenerate case (though perfectly suited to the Ostrogradski canonicalization procedure with second class constraints).[34] The higher derivative terms at these low orders arise only from coordinate and gauge choices,[35] but there is little reason to doubt that non-degenerate higher derivative terms will appear at higher order.

Another important case where perturbative constraints have not been considered, but should be, is the case of gravity as a low energy limit of string theory. Einsteins's equations and the corrections to arbitrarily high orders (in the slope parameter) can be obtained from non-linear sigma models (*e.g.* de Alwis[7]).

The first order corrected gravitational action in D dimensions is

$$S \propto \int d^D x \sqrt{-g}\left[R - \frac{\alpha'}{4}\left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - \nabla^2 R\right) + (\text{matter terms}) + O(\alpha'^2)\right] \qquad (2-50)$$

where α' is the slope parameter. As in the preceding case, to the extent that the zeroth order theory (Einstein gravity) is a good approximation of nature, higher order terms produced by this method should only be perturbative corrections; they should not completely alter the dynamics of the system. The perturbative constraints must be applied for consistency.

It is important to note in this case that the large non-local theory which Einstein gravity and the constrained higher order terms approximates well is not string theory itself. It is a non-local, low energy effective theory that is derivable from string theory and is appropriate in cases where Einstein gravity is also a good approximation (though not as good as the non-local low energy effective theory). Neither is appropriate in regions of very high curvature or near singularities. The analog of the intermediate non-local theory for the case of electrodynamics would be the Wheeler-Feynman theory, which falls between full field theoretic electrodynamics and slowly moving, non-radiating, Lorentz-force-law motion. Just as the Wheeler-Feynman theory is not accurate at high energies (comparable to the electron mass), neither general relativity nor general relativity plus string corrections will not be accurate at high curvatures.

When the higher derivative corrections arise from quantum effects, then the above argument does not hold. One may or may not choose to apply the perturbative constraints. But one should be aware that the higher derivative theory without perturbative constraints is dramatically different from Einstein gravity, while the same higher derivative theory with perturbative constraints is a true perturbative correction. If there is any reason to believe that the quantum corrections will not radically alter the behavior of the system, then the perturbative constraints must be applied. (The same holds for Lovelock gravities, which, while strictly speaking not higher derivative theories, have some solutions that are close to Einstein gravity and others that are far from Einstein gravity. One must throw away the dramatically different solutions, i.e. impose the perturbative constraints, if the corrections to Einstein gravity are intended to be small.)

The perturbatively constrained system has qualitatively different properties than the unconstrained higher derivative theory. The renormalizability gained from the higher derivatives is lost once the constraints are applied. The extra particles (degrees of freedom) present in the unconstrained theory do not exist in the constrained theory, since any solution containing them is non-analytic in the expansion parameters.

For higher order terms $\propto R^2$, all vacuum solutions to the Einstein action are still vacuum solutions, and so just as for the case of cosmic strings and the unforced

Dirac electron, the perturbative constraint is just the old equation of motion. This has the effect that the new equations of motion ignore the R^2 piece entirely (though again, when coupled to matter, this could easily change). Dramatically different solutions, such as those offered in Starobinsky-type inflation,[19] are excluded from the realm of acceptable solutions. For the additional non-local, higher derivative terms arising from quantum corrections, the first order terms contribute non-trivially even in vacuum, but do not dominate the evolution.[36] In the case of Lovelock gravities, the de Sitter-like and anti de Sitter-like solutions[27] are disallowed as acceptable spherically symmetric solutions.

VI. Summary

Higher derivative theories occur in various places throughout theoretical physics, usually as a correction term to a standard, lower derivative theory. In particular, they arise in the context of theories of gravity and of cosmic strings. Though the higher derivative theories are mathematically self-consistent, there are distinctive features of unconstrained higher derivative theories that set them apart from similar lower derivative theories. There are more degrees of freedom, associated with new solutions called "runaways", qualitatively different from those of a related lower derivative theory. There is no lower energy bound. In the case of field theories, these features can cause the problems of "ghost" fields and loss of unitarity.

There is a natural way to constrain many higher derivative theories and save them from the above problems. It applies in cases where the higher derivative terms are associated with a small, perturbative, expansion parameter. The method, called the method of perturbative constraints, is to exclude solutions that have no Taylor expansion in that small expansion parameter. The effect is to throw away all the runaway, negative energy solutions. Without the perturbative constraints, higher order terms in the expansion contribute as much as the lower order terms, not commensurately less.

Higher derivative theories that are expansions of a non-local theory require these perturbative constraints to give the same results as the full non-local theory. The perturbative constraints are actually present implicitly in the full theory, but they must be included explicitly for any finite expansion. Important examples of this case, where the perturbative constraints should be (but have not been) applied include higher order corrections to general relativity from string theory, and to cosmic strings from the original gauge theory from which they arise. Non-locality is a common feature in low energy effective theories, and it not at all necessarily present in the full theory from which they are derived.

Higher derivative theories which are not necessarily a truncated version of an infinite series, but can still be viewed as corrections to a valid lower derivative theory, can also reap the benefits of the method of perturbative constraints. The constrained theory will resemble the original, lower order theory in its solutions and number of degrees of freedom, and will have a lower energy bound, all of which one would hope for an a perturbative expansion. The constraints are necessary if the perturbative higher derivative corrections are to produce perturbative solutions. The unconstrained version would have all of the associated problems of higher derivative theories, and the higher derivative "correction" would completely dominate the behavior of the solutions, complete with negative energy modes.

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Chapter 3

The Stability of Flat Space, Semiclassical Gravity, and Higher Derivatives

Flat space is shown to be perturbatively stable, to first order in $\hbar$, against quantum fluctuations produced in semiclassical (or $1/N$ expansion) approximations to quantum gravity, despite past indications to the contrary. It is pointed out that most of the new "solutions" allowed by the semiclassical corrections do not fall within the perturbative framework, unlike the effective action and field equations which generate them. It is shown that excluding these non-perturbative "pseudo-solutions" is the only self-consistent approach. The remaining physical solutions do fall within the perturbative formalism, do not require the introduction of new degrees of freedom, and suffer none of the pathologies of unconstrained higher derivative systems. As a demonstration, a simple model is solved, for which the correct answer is not obtained unless the non-perturbative pseudo-solutions are excluded. The presence of the higher derivative terms in the semiclassical corrections may be related to non-locality.

I. Introduction

Our everyday experience tell us that flat space is stable (or at least very metastable) against small perturbations in matter or curvature. Theoretically this has been shown to be the case for classical general relativity (and matter obeying the dominant energy condition) by the proof of the positive energy theorem.[37] It has been suggested, however, that quantum corrections to classical general relativity might change this result. Issues of stability in quantum mechanics can be trickier and more subtle than in classical mechanics, but nonetheless there have been several strong indications of the instability of gravity when coupled to quantum fields.

Attempts to examine quantum effects on gravity have been made using semiclassical and $1/N$ expansion approximations. In semiclassical approximations, it appeared that the gravitational curvature could either grow very large on a time scale of order of the Planck time or generate large scale radiation production with this frequency.[3,4] In $1/N$ approximations, where gravity is quantized as well as the matter fields, it appeared that the expectation value of the energy could be lowered from that of flat space, and that the gravitational propagator contained tachyonic modes, both of which imply instability.[5] These calculations are particularly disturbing because they hint that flat space is unstable against quantum perturbations, in contradiction with our everyday experience. Because the field equations for the semiclassical system (for our purposes, both $1/N$ and semiclassical methods may be treated on an equal footing) contains terms that are higher derivative than in the lowest order (classical Einstein) system, the solution space is potentially larger than in the lowest order case. New solutions arising only from the presence of higher derivatives describe the instabilities found above.

Recent work, however, sheds new light on the relationship between the higher derivative terms and the full, non-perturbative system from which they arise. The older analyses[3,4,5] begin by assuming that it is appropriate to perturbatively expand the effective action describing geometry in the presence of matter fields (and so also the field equations) in powers of $\hbar$. In the case of gravity, to lowest (zeroth) order, the effective action is just the classical Einstein-Hilbert action. The first order correction contains terms that are second order in time derivatives (see equation (3 - 1)). These give rise in their field equations to terms that are fourth order in time derivatives, and therefore entirely new families of solutions not present in the lowest order, second order differential equation. Most of these new solutions are non-analytic in $\hbar$ (as $\hbar \rightarrow 0$), and so, if used, violate the initial perturbative ansatz. In fact, neither the expanded action nor the expanded field equations, if evaluated at a new, non-perturbative solution, remain perturbative expansions in $\hbar$. To be internally consistent, the solution space must be restricted to only solutions perturbatively expandable in $\hbar$. It had been hoped, or perhaps tacitly assumed, that, despite this inconsistency, the apparently new solutions would give insight to the behavior of the solutions of the full, non-perturbative effective action. While this cannot be explicitly ruled out, a more likely explanation is that the higher derivative terms are not related to non-perturbative behavior of solutions of the full action, but instead arise from perturbatively expanding a non-local expression. This is a common feature of perturbatively expanded non-local actions, as the examples below will show. In these cases the higher derivative terms that arise do not correspond in any way to non-perturbative behavior of the full action, but they would give rise to false, non-perturbative "pseudo-solutions" if the perturbative ansatz were abandoned half way through the calculation. These pseudo-solutions are never perturbatively expandable in $\hbar$, even in the case where the action and field equations are perturbative expansions. A self-consistent method for restricting solutions to remain within the perturbative framework is presented below.

Even if the non-dynamical higher derivatives appear for reasons other than non-locality, the non-perturbative pseudo-solutions must still be excluded for self-consistency, if the action itself is a perturbative approximation. Whatever the full quantum theory of gravity may be, it is expected to possess a low energy effective action, of which the first few terms of the truncated perturbative expansion would be semiclassical gravity. By remaining within the perturbative framework, although non-perturbative information is lost or hidden, at least self consistency is maintained. If one were to abandon the perturbative ansatz once new solutions were found outside the domain of formal perturbative expansions, false conclusions could easily be drawn, and because self-consistency would be lost, the relationship between the effective theory and the full theory would be lost as well.

II. Quantum Corrections To Gravity

Some quantum corrections to gravity can be calculated without the full quantum theory. One approach is the semiclassical method, in which purely classical gravity is driven by the expectation value of quantum matter. This approximation should be valid in many interesting cases, where the gravitational part of the wave function of spacetime behaves strongly semiclassically, but quantum effects are important for the matter fields. Important examples are the back reaction of Hawking radiation on the metric of a large evaporating black hole, and the the back reaction of particles created in the transition from an inflationary era to a radiation dominated era. The semiclassical approximation would be expected to break down in situations where the effect of the quantum matter on gravitation is to drive it into a regime of high (Planck scale) curvature, such as the final stages of an evaporating black hole, or at very early times in the universe. Solutions produced by the semiclassical approach that make predictions in such a regime should not be considered physical results.

One quite general approach to semiclassical approximations of quantum gravity was implemented by Horowitz[4], using Wald's stress energy axioms[38] to constrain the form of the semiclassical field equations. Another method, even more general in some respects, is the $1/N$ approximation of Hartle and Horowitz[5], which quantizes gravity coupled to N matter fields, and then examines the large N limit. The first term in the $1/N$ expansion gives a semiclassical-like field equation, but higher order corrections are (in principle) calculable as well, a feature lacking when gravity is kept strictly classical.

All of these approaches to quantum corrections to gravity share common features. The effective field equations are higher order in time derivatives than the classical equation, and the higher order terms have small coefficients (proportional to $\hbar$). If taken seriously, higher derivatives mean that twice as much initial data must be specified to evolve the system forward in time, or, in the variational formulation, twice as much data must be specified on the boundaries. In the initial data formulation, not only must the metric and its first derivative be specified, but also the second and third time derivatives. In the variational formulation, not only must the metric be specified on the boundary (or boundaries), but also its first derivative. It would make semiclassical gravity very different from almost all other physical dynamical theories, which are almost always second order in time. Furthermore, as higher order corrections are considered when the gravitational field is also quantized, terms proportional to higher powers and higher derivatives of curvature are expected. This would have the bizarre effect of requiring more and more initial data to be specified as terms of (supposedly) less and less importance are considered.

For the moment, let us consider only the first order corrections, which make the field equations fourth order in time. The effective action takes the form

$$S_{\text{eff}} = \frac{1}{16\pi G}\int d^4x\sqrt{g}\Big(-2\Lambda + R + \alpha R^2 + \beta R_{ab}R^{ab} + \gamma(\text{terms non-local in curvature})\Big) + (\text{surface terms}) \qquad (3\text{ - }1)$$

where α, β and γ are all proportional to $\hbar$ (the terms non-local in curvature in the action lead to purely local terms in the field equations; they should not be confused with non-locality in the equations of motion discussed at length below). We use the conventions $c = 1$, $\eta_{ab} = (-+++)$, $R^l{}_{man} = \partial_a\Gamma^l{}_{mn} + \ldots$, and $R^l{}_{mln} = R_{mn}$. In the semiclassical case were found tachyonic and exponentially growing fluctuations, both of which strongly indicate an instability of flat space.[4] In the $1/N$ expansion were found fluctuations of negative energy and also tachyonic poles in the gravitational propagator.[5] In all such cases, choosing certain values of some parameters could lessen some of the unstable behavior, but for no combinations could the instabilities be made to vanish (this is true even if the probably unphysical Landau ghost discussed by Hartle and Horowitz is discounted as an instability[5]).

An insufficiently stressed property of the solutions contributing to the above instabilities is that, despite the fact that both the effective action and the field equations governing the quantum corrections are perturbative expansions in $\hbar$, most of the solutions are not perturbatively expandable in $\hbar$ (*i.e.* not analytic functions of

$\hbar$ as $\hbar \to 0$). This can be seen from the field equations, which for a metric g, are roughly

$$\ddot{g} = f^{\mathrm{cl}}(g,\dot{g}) + \hbar f^{1}(g,\dot{g},\ddot{g},\dddot{g},g^{(4)}) \qquad (3 - 2)$$

where dots represent time derivatives. In order to invert this equation for the highest derivative of the metric, it is necessary to divide by $\hbar$. On dimensional grounds, the natural time scale is the Planck time, $t_{\mathrm{pl}} = (G\hbar)^{1/2}$, and solutions generally behave as functions of t / t_{pl}. We are faced with the prospect of non-perturbative solutions to a perturbative expansion. The presence of still higher derivatives in still higher order corrections makes the situations look even stranger.

Short of giving up completely, on the grounds that semiclassical gravity might be irredeemably inconsistent, there are two directions to proceed. The first is to accept the new non-perturbative solutions as valid. This has been the more popular path historically. There is some hope that the non-perturbative solutions are actually giving some (unexpected) insight into the non-perturbative behavior of the full quantum gravitational theory. There is little motivation for this, since semiclassical gravity is only expected to approximate a perturbative expansion of the full theory. In any truncated perturbative expansion, non-perturbative behavior has necessarily already been lost.

The second path is to take the perturbative expansion seriously and exclude all solutions not perturbatively expandable in $\hbar$ as fictitious. This is the approach we put forward in this paper. The primary advantage of this approach is self-consistency: the effective action is a formal perturbative expansion, the field equations are formal perturbative expansions, and so should the solutions be. Furthermore, the action and the field equations lose their interpretation as a perturbative expansion if evaluated at non-perturbative extrema. That is, the "higher order" terms are not higher order when evaluated on a non-perturbative "pseudo-solution". Unless the perturbative expansion holds at the extrema, there is no reason the effective action should be expected to approximate the full action in any sense, evaluated near the extrema. The applicability of perturbation theory to the stability of action-based physical systems is discussed in Appendix A. The second benefit to taking the perturbative expansion seriously is that the solution space does not grow as the perturbative order is increased. A result of Jaén, Llosa, and Molina[13] shows that, to any order, the same amount of initial data will suffice for all solutions analytic in the perturbative expansion parameter of any system of the form

$$L = \frac{1}{2}\sum_{a=1}^{N} m_a \dot{q}_a^2 + \sum_{l=0}^{n} \varepsilon^l V_l\left(q, \frac{dq}{dt}, \ldots, \frac{d^l q}{dt^l}\right) + O\left(\varepsilon^{n+1}\right) \tag{3 - 3}$$

where ε is the perturbative expansion parameter and m_a is the mass of particles a = 1, ..., N, and the matrices $\partial^2 V_l \,/\, \partial q_a^{(l)} \partial q_b^{(l)}$ are regular. Their proof demonstrates that all but N of the momenta of this system cannot be inverted within the formalism of perturbative expansions, corresponding to the presence of constraints, which are shown to be second class constraints. The constrained system has the same number of degrees of freedom for any n, including $n = 0$. This result can be generalized to more complicated systems, as will be done below for linearized gravity, which has additional fields present in the first order correction not present in the lowest order action.

To reiterate, the advantage of taking the perturbative expansion seriously is self-consistency: 1) the initial action and field equations are formal perturbative expansions and now the solutions are also formal perturbative expansions; 2) the number of degrees of freedom of the system is fixed and does not depend on the order to which the expansion is taken; 3) the system plus the constraints necessary to exclude the non-perturbative pseudo-solutions is strongly equivalent (in the sense of Dirac constrained systems) to a second order system, and thus has none of the pathologies of unconstrained higher derivative theories. The consequences of losing self-consistency are the appearance of spurious solutions to the truncated series, not related to any solutions of the full action. Theses spurious solutions occur even in simple examples (as shown below), and must be excluded if solutions to the truncated expansion are to approximate solutions to the full action.

Even if the more consistent, perturbative direction is taken, one might still reasonably ask why the extra solutions that must be excluded arise at all. What is the purpose of the higher derivatives in the effective action and field equations? There may be several answers to this question, but an answer common to many theories based on effective actions is that the higher derivatives come from non-locality. This is discussed next.

III. Non-Locality, Perturbative Expansions with Higher Derivatives

Non-locality is is a feature often displayed in theories based on effective actions, *i.e.* a theory made simpler by integrating out some subset of its degrees of freedom. Effective actions describe theories with "action at a distance" since some fields have been deprived of their dynamical status. One example of a theory described by an effective action is semiclassical electrodynamics, where the electromagnetic fields are classical but the quantum nature of the matter fields are retained.[39] Another is the Wheeler-Feynman theory of classical electrodynamics, in which electrons interact non-locally via half retarded/half advanced potentials, without dynamical electromagnetic fields.[31] Since Einstein gravity is non-renormalizable, it is likely that it is not a fundamental theory but, rather, the low energy limit of an effective theory based on some larger, fundamental "theory of everything" (perhaps string theory). The effective low energy theory predicted by superstrings will be discussed below.

Non-local theories for which the non-locality is regulated by a small, dimensionful parameter can produce higher derivatives when perturbatively expanded in that parameter. For instance, a function that is non-local in time, such as $x(t+\varepsilon t')$, can be expanded in powers of ε. For example

$$x(t+\varepsilon t') = \sum_{n=0}^{\infty} \frac{(\varepsilon t')^n}{n!} \frac{d^n x(t)}{dt^n} \tag{3 - 4}$$

In this way an infinite sum of individually local, higher derivative terms can represent a non-local expression. The full non-local theory may or may not contain behavior usually associated with purely higher derivative theories (*e.g.* additional degrees of freedom, lack of a lowest energy state; see Eliezer and Woodard[14] for a lucid presentation of higher derivatives and non-locality). If such an expansion is used for a non-local action, any finite truncation of the sum may behave very differently from the full theory. In particular, the number of degrees of freedom of the truncated sum appears to depend of the degree of truncation, whereas the number of degrees of freedom of the full theory is fixed. The only solution to this problem is to agree that for any finite truncation one will only examine consistent perturbative solutions. Such an agreement does not deny the existence of possible non-perturbative behavior of the full theory, but it does acknowledge that such behavior is inaccessible in the perturbative expansion already performed. At the very least, non-local theories demonstrate how higher derivatives may appear in an approximate theory and not represent dynamical degrees of freedom.

A simple example of a non-local theory can help develop some intuition for the subject. The model is of a non-local harmonic oscillator (for a fuller treatment, including quantization, see the previous chapter). The potential of this harmonic oscillator is non-local in the sense that it depends not only on the position of the spring at a specific instant, but also on the position in the past and future, with heavier weighting of times near the present . This model simply displays the effects of non-locality and the appearance of higher derivatives in a perturbative expansion, and it has the important advantage of being exactly soluble. The model's equation of motion is

$$\ddot{x}(t) = -\omega_0^2 \int_0^\infty ds e^{-s} \frac{1}{2}[x(t+\varepsilon s) + x(t-\varepsilon s)] \qquad (3-5)$$

where $\varepsilon\omega_0 < 1$. In the limit $\varepsilon \to 0$, we regain the simple harmonic oscillator equation $\ddot{x} = -\omega_0^2 x$. The two parameter family of exact solutions is given by

$$x = A\cos(\omega t + \phi) \qquad (3-6)$$

where A and ϕ depend on the initial conditions and

$$\omega^2 = \omega_0^2\left(\frac{1}{2} + \frac{1}{2}\sqrt{1+4\varepsilon^2\omega_0^2}\right)^{-1} = \omega_0^2\left(1 - \varepsilon^2\omega_0^2 + 2\varepsilon^4\omega_0^4 + \ldots\right) \qquad (3-7)$$

is the new effective frequency due to non local effects.

One may also solve the system perturbatively and compare the result with the exact solution. Since both the equation of motion and the general solution are perturbatively expandable in ε , there should be no obstacles. The equation of motion becomes

$$\ddot{x} = -\omega_0^2(x + \varepsilon^2\ddot{x} + \varepsilon^4 x^{(4)} + \varepsilon^6 x^{(6)} + \ldots) \qquad (3-8)$$

There appears to be an arbitrarily high number of degrees of freedom due to the infinite sum of higher derivatives. In fact, we know that the exact solution has only 2 arbitrary parameters, so all other degrees of freedom must be excluded implicitly in demanding that the sum converge. If we truncate at any finite order, though, we lose the implicit constraints, and we must then explicitly exclude non-perturbative solutions. Truncating (3 - 8) at ε^0 or ε^2 and solving gives no trouble because the equation of motion remains second order and gives the correct answers

$$\begin{aligned} \varepsilon^0: &\quad x = A\cos(\omega_0 t + \phi) \\ \varepsilon^2: &\quad x = A\cos(\omega_2 t + \phi) \end{aligned} \qquad (3 - 9)$$

to the appropriate order in ε, where $\omega_2^2 = \omega_0^2\left(1 - \varepsilon^2\omega_0^2 + \ldots\right) = \omega^2 + O(\varepsilon^4)$ is an easily calculable function of ε and ω_0. Truncating (3 - 8) at higher orders, however, gives extra "pseudo-solutions" occur that are not perturbatively expandable in ε.

$$\begin{aligned} \varepsilon^4: &\quad x = A\cos(\omega_4 t + \phi) + B\cos(\gamma t + \psi) && \gamma \sim \frac{1}{\varepsilon}\frac{1}{\varepsilon\omega_0} \\ \varepsilon^6: &\quad x = A\cos(\omega_6 t + \phi) + B_+\cos(\gamma_+ t + \psi_+) && \\ &\qquad\qquad + B_-\cos(\gamma_- t + \psi_-) && \gamma_\pm \sim \frac{1}{\varepsilon}\frac{1}{\sqrt{\pm\iota\varepsilon\omega_0}} \end{aligned} \qquad (3 - 10)$$

and so on, where $\omega_{2n}^2 = \omega^2 + O(\varepsilon^{2n+2})$ is a calculable function of ε and ω_0 in each case.

Thus, this simple model is an explicit example of how abandoning the perturbative formalism for the solution simply gives the wrong answer. Retaining the perturbative formalism (that is, excluding, by the appropriate constraints, all non-perturbative results) gives the correct answer, to any order. We see that when the order of derivatives grows with the order of expansion, it is an obvious symptom of non-locality. It alerts us that the higher derivatives do not represent dynamical degrees of freedom but are an artifact of the expansion. Keeping only perturbative solutions is the only self-consistent path available.[40]

Solving for all exact solutions of the truncated expansion and then discarding those not perturbatively expandable, while a valid procedure, is computationally wasteful and may not always be possible. A more feasible prescription is to solve the equations of motion while remaining, at every step, strictly within the perturbative formalism.

"Strictly within the perturbative formalism" means that, in solving the field equations, all expressions must be polynomials (formal expansions) in the perturbative constant, up to the specified order of the truncation. Only operations which preserve the formal expansion are permitted. One may consider the perturbative expansion parameter to be not an ordinary number, but an abstract object with no multiplicative inverse (once the perturbative order is set). Division by terms containing the perturbative constant is *forbidden* (though multiplying by a reciprocal, *if it exists*, is allowed), once the perturbative order is set. Note that the strictly perturbative formalism implies that if $f(x)+\varepsilon g(x)=0+O(\varepsilon^2)$, and f and g are both zeroth order in ε, then both f and g must vanish independently. Note also that the vanishing of the product of two terms does not guarantee that either must vanish (*e.g.* $\varepsilon \times \varepsilon = 0 + O(\varepsilon^2)$). Algebraically speaking, the system is a commutative ring with zero divisors, where the role of zero element is played by $O(\varepsilon^{N+1})$.[13]

To make these ideas more concrete, we will solve the example system above by this method, truncated to powers of ε^4. The equation of motion is

$$\ddot{x} + \omega_0^2 x + \varepsilon^2 \omega_0^2 \ddot{x} + \varepsilon^4 \omega_0^2 x^{(4)} = O(\varepsilon^6) \quad . \tag{3 - 11}$$

Dividing by ε^4 is forbidden if the equation is to remain a perturbative expansion to $O(\varepsilon^4)$. Instead we multiply by ε^4

$$\varepsilon^4 \ddot{x} + \varepsilon^4 \omega_0^2 x = O(\varepsilon^6), \tag{3 - 12}$$

take 2 time derivatives

$$\varepsilon^4 x^{(4)} + \varepsilon^4 \omega_0^2 \ddot{x} = O(\varepsilon^6), \tag{3 - 13}$$

and substitute back into (3 - 11) to get

$$\ddot{x}(1 + \varepsilon^2 \omega_0^2 - \varepsilon^4 \omega_0^4) + \omega_0^2 x = O(\varepsilon^6) \quad . \tag{3 - 14}$$

We are still forbidden to divide by any expression containing ε, but we may still multiply by the reciprocal if it exists. Since

$$(1 + \varepsilon^2 \omega_0^2 - \varepsilon^4 \omega_0^4)(1 - \varepsilon^2 \omega_0^2 + 2\varepsilon^4 \omega_0^4) = 1 + O(\varepsilon^6) \quad , \tag{3 - 15}$$

the final form of the equation of motion is

$$\ddot{x} + \omega_0^2 (1 - \varepsilon^2 \omega_0^2 + 2\varepsilon^4 \omega_0^4) x = O(\varepsilon^6). \tag{3 - 16}$$

Compare this with (3 - 7) to see that this gives the correct answer to the full equation of motion (to order ε^4), and compare with the first line of (3 - 10) to see that this also agrees with the method of solving for all solutions and afterwards excising all non-perturbative pseudo-solutions. That we are not missing any perturbative solutions is guaranteed by (3 - 3).

IV. Quantum Corrections To Gravity Revisited

We may now consider these ideas in the specific context of quantum corrections to gravity. Whatever properties the full quantum theory of gravity may have, it is expected to possess a low energy effective action that can be expanded in powers of the Planck time, $t_{\rm pl} = (\hbar G)^{1/2}$, and there is no reason to suspect that the expansion ends at any finite order. For example superstrings predict an effective low energy theory with an infinite expansion given by[7]

$$S = \frac{1}{2\alpha'} \int d^d x \sqrt{g} \left(R - \frac{\alpha'}{4} R_{abcd} R^{abcd} + \frac{\alpha'}{4} \nabla^2 R + (\text{matter}) + O(\alpha'^2) \right) \qquad (3 - 17)$$

at tree level, where α' is the slope parameter, with dimensions of $l_{\rm pl}^2$. On dimensional grounds, higher order corrections will be accompanied by higher powers of curvature and its derivatives, giving higher and higher time derivatives. Einstein gravity itself is non-renormalizable, and so makes no predictions concerning the form of higher order terms in the expansion. Nevertheless, to whatever extent semiclassical, $1/N$, or any other approximations scheme is to agree with predictions of the full theory, it must be treated as giving the first few terms of a larger expansion. Since non-locality is a common feature of effective actions, it is quite plausible that all higher derivative terms arise from the perturbative expansion of non-locality, and, therefore, that the non-perturbative pseudo-solutions should be excluded. Still, even if the non-dynamical higher derivatives appear for reasons other than non-locality, the non-perturbative pseudo-solutions must still be excluded for self-consistency, if the action itself is a perturbative approximation. Information of non-perturbative solutions has already been lost in making the perturbative approximation of the action and field equations. It is impossible to tell whether the non-perturbative pseudo solutions are at all related to any lost non-perturbative solutions, but excluding them is at least self-consistent.

The effects of excluding the pseudo-solutions are several. First, we show that there are no new degrees of freedom or fields. The most general higher derivative, semiclassical corrections found by Horowitz[4] can be written most concisely in terms of the Fourier transform

$$S_{\text{eff}} = \int \frac{d^4k}{(2\pi)^4}\sqrt{-g}\left\{\kappa^{-2}R + a\ln\left(\frac{k^2}{\mu^2}\right)C^*_{abcd}C^{abcd} + \left[b\ln\left(\frac{k^2}{\mu^2}\right)+\alpha\right]R^*R\right\} + O(\hbar^2) \tag{3 - 18}$$

where * denotes complex conjugation, and a,b, and α are all proportional to $\hbar$, and their exact values depend on which matter fields couple to gravity and which regularization scheme is chosen in the process of renormalization. C_{abcd} is the

Weyl tensor. Following Stelle,[15] we decompose the linearized metric into transverse traceless, transverse, and longitudinal components.

$$
\begin{aligned}
h_{ij} &= h_{ij}^{\mathrm{TT}} + h_{ij}^{\mathrm{T}} + k_i\xi_j + k_j\xi_i \\
h_{i0} &= h_{i0}^{\mathrm{T}} + k_i\xi_0 + k_0\xi_i \\
\tilde{h}_{00} &= h_{00} - 2k_0\xi_0
\end{aligned}
\qquad (3\text{ - }19)
$$

where

$$
\begin{aligned}
h^{\mathrm{T}} &= h_{ii} - \vec{k}^{-2}k_ik_jh_{ij} \\
h_{ij}^{\mathrm{T}} &= \frac{1}{2}\left(\delta_{ij}h^{\mathrm{T}} - \vec{k}^{-2}k_ik_jh^{\mathrm{T}}\right) \\
h_{ij}^{\mathrm{TT}} &= (\delta_{ik} - \vec{k}^{-2}k_ik_k)(\delta_{lj} - \vec{k}^{-2}k_lk_j)h_{kl} \\
&\quad -\frac{1}{2}(\delta_{ij} - \vec{k}^{-2}k_ik_j)(\delta_{kl} - \vec{k}^{-2}k_kk_l)h_{kl} \\
h_{i0}^{\mathrm{T}} &= h_{i0} - \vec{k}^{-2}(k_lk_ih_{0l} + k_lk_0h_{il} - \vec{k}^{-2}k_ik_kk_jk_0h_{jk}) \\
\xi_i &= \vec{k}^{-2}(k_jh_{ij} - \frac{1}{2}\vec{k}^{-2}k_kk_jk_ih_{kj}) \\
\xi_0 &= \vec{k}^{-2}(k_ih_{0i} - \frac{1}{2}\vec{k}^{-2}k_ik_jk_0h_{ij})
\end{aligned}
\qquad (3\text{ - }20)
$$

and h_{ij}^{TT}, h^{T}, h_{i0}^{T}, and $\tilde{\mathrm{h}}_{00}$ are invariant under the transformation $h_{mn} \to h_{mn} + k_{(m}\eta_{n)}$ for arbitrary η_n. Inserting this decomposition into the linearized action gives

$$
\begin{aligned}
S_{\mathrm{eff}}^{\mathrm{lin}} = \int \frac{d^4k}{(2\pi)^4}\Big\{ &\frac{1}{2}h_{ij}^{\mathrm{TT}*}(\kappa^{-2} - fk^2)k^2h_{ij}^{\mathrm{TT}} - \frac{1}{4}h^{\mathrm{T}*}[\kappa^{-2} + (\frac{1}{3}f + 4g)k^2]k^2h^{\mathrm{T}} \\
&+ \mathrm{Re}\, h^{\mathrm{T}*}[\kappa^{-2} + (-\frac{1}{3}f + 2g)k^2]\tilde{h}_{00} - \tilde{h}_{00}^{*}(\frac{1}{3}f + g)\vec{k}^4\tilde{h}_{00} \\
&- h_{0i}^{\mathrm{T}*}(\kappa^{-2} - fk^2)\vec{k}^2h_{0i}^{\mathrm{T}}\Big\} + O(\hbar^2)
\end{aligned}
\qquad (3\text{ - }21)
$$

where $f = a\ln(k^2/\mu^2)$, $g = b\ln(k^2/\mu^2) + \alpha$, and the field equations are given by

$$\delta S_{\rm eff}^{\rm lin} = 0 + O(\hbar^2)\ . \tag{3 - 22}$$

Since this is independent of ξ_m, the ξ_m are the natural gauge variables of the linearized system. In the classical limit, $f = g = 0$, and the reader may verify that only the $h_{ij}^{\rm TT}$ are dynamical in this limit. Following the same steps as for the simple model above, multiply (3 - 22) by $\hbar$ to get

$$\hbar \delta S_{\rm eff}^{\rm lin} = 0 + O(\hbar^2) \tag{3 - 23}$$

which is equivalent to

$$\begin{aligned} &\hbar \Box h_{ij}^{\rm TT} = 0 + O(\hbar^2) \\ &\hbar h^{\rm T} = \hbar \tilde{h}_{00} = \hbar h_{i0}^{\rm T} = 0 + O(\hbar^2) \end{aligned} \tag{3 - 24}$$

Recall that division by $\hbar$ is not allowed if we are to remain at the same order. Since all corrections to the field equations are of the form of (3 - 24), they also vanish (to this order). The only solutions to the linearized field equations that are perturbatively expandable in $\hbar$ are the same as the solutions to the classical equations, but now to one higher order in $\hbar$: $\Box h_{ij}^{\rm TT} = 0 + O(\hbar^2)$. There cannot be any other solutions perturbatively expandable in $\hbar$ because of the second class constraints associated with the momenta and time derivatives of $h^{\rm T}$, $\tilde{h}_{00}$, and $h_{i0}^{\rm T}$ and remaining within the perturbative formalism. The momenta cannot be inverted within the confines of strict perturbation theory, signaling the presence of primary constraints. These constraints, along with their associated secondary constraints, do not commute, *i.e.* they are second class. The result is that $h^{\rm T}$, $\tilde{h}_{00}$, and $h_{i0}^{\rm T}$are not dynamical fields. The only field degrees of freedom are those of the graviton(h_{ij}^{TT}). This should not be too surprising in the context of Stelle's analysis. The apparently new degrees of freedom found there corresponded to particles with masses inversely proportional to the Planck length. Any excitations of those false degrees of freedom would result in frequencies also of order the Planck scale, corresponding to solutions that diverge as $\hbar \to 0$.

It is the fictitious degrees of freedom excised above that are responsible for indications of the instability of flat space. Previous analyses of the stability of flat space found "solutions" to the semiclassical equations with behavior $\sim t / t_{\mathrm{pl}}$. For instance, Horowitz and Wald find modes of real or imaginary frequency $(48\pi\alpha)^{-1/2}$ (where $\alpha \propto \hbar$ is defined in equation (3 - 1)) which lead to instabilities either from runaway solutions or enormous radiation production.[3] Below we will reanalyze in detail the energy analysis of Hartle and Horowitz[5] in the perturbative formalism. Generalizing these techniques to other analyses of the stability of flat space time is straightforward.

The energy analysis of Hartle and Horowitz[5] computes the minimum energy among all states for which the expectation value of the metric is a given stationary geometry satisfying the constraints of the system. The answer may be expressed in terms of the effective action by[41]

$$E[g] = -\frac{S_{\text{eff}}[g]}{T} \tag{3 - 25}$$

where T is the time integrated over in evaluating S_{eff} and g is the stationary geometry. The original analysis found that this quantity can be made negative for some deformations of the metric (the energy vanishes for flat space), indicating an instability of flat space. When the analysis is re-performed within the formalism of perturbation theory, as will be seen, no such indications are found.

The linearized effective action can be written

$$S_{\text{eff}}[h] = -\frac{1}{2}\int \frac{d^4k}{(2\pi)^4}(h^{ab})^*(\dot{G}_{ab} - f\dot{A}_{ab} - g\dot{B}_{ab}) + O(\hbar^2) \tag{3 - 26}$$

where a single dot denotes the linearized approximation (not a time derivative), and

$$\begin{aligned}
\dot{A}_{ab} &= 2k^2\dot{G}_{ab} + \frac{2}{3}(k^2\eta_{ab} - k_a k_b)\dot{R} \\
\dot{B}_{ab} &= 2(k^2\eta_{ab} - k_a k_b)\dot{R} \\
\dot{G}_{ab} &= \frac{1}{2}[k^2 h_{ab} - 2k^c k_{(a}h_{b)c} + k_a k_b h^c{}_c + \eta_{ab}(k^c k^d h_{cd} - k^2 h^c{}_d)] \\
\dot{R} &= -\dot{G}^a{}_a = k^2 h^a{}_a - k^a k^b h_{ab}
\end{aligned} \tag{3 - 27}$$

We must also, however, include the new second class constraints documented in the previous section, *i.e.* that there are no new degrees of freedom. The constraints (3 - 23) can be summarized covariantly as

$$\hbar\dot{G}_{ab} = 0 + O(\hbar^2) \tag{3 - 28}$$

and can also be derived by putting the system in canonical form, but retaining the perturbative expansion formalism. There the momenta cannot be inverted within the perturbative formalism, which leads to new primary constraints[13] (in addition to the usual first class constraints of general relativity), which in this case is (3 - 28). Its accompanying secondary constraint is the time derivative of (3 - 28), and the two constraints are second class (*i.e.* they do not commute), reflecting the fact that the number of field degrees of freedom is smaller than is expected in a higher derivative action (in contrast to the still present first class constraints of general relativity, which signify gauge freedom). Both $f\dot{A}_{ab}$ and $g\dot{B}_{ab}$ are proportional to $\hbar\dot{G}_{ab}$, and so vanish (to this order), leaving the effective action equal to the classical action. The action, field equations, and usual first class constraints are all the same as the classical case (but now to higher order), and so the energy functional is also the same.

$$E[h_{ij}] = \frac{1}{4}\int \frac{d^3k}{(2\pi)^3}(\vec{k}^2\left|h_{ij}^{\mathrm{TT}}\right|^2) \geq 0 + O(\hbar^2) \quad . \tag{3 - 29}$$

Thus, remaining in the perturbative framework guarantees that the energy of flat space cannot be lowered perturbatively, to first order in $\hbar$.

The same constraint, (3 - 28), applies to all semiclassical expansions about flat space and vacuum matter (it does not apply, for instance, to semiclassical expansions in the presence of a cosmological constant, where, for quantum corrections to de Sitter space, we would have $\hbar G_{ab} = \hbar\Lambda g_{ab} + O(\hbar^2)$). Any examination of corrections to other gravitational behavior must also take this constraint into account. For instance, the structure of the graviton propagator, which without (3 - 28) would have tachyonic poles at planck-like frequencies,[5] behaves exactly as the classical propagator.

Excluding the non-perturbative pseudo-solutions by no means proves that flat space is stable against quantum effects of gravity, but, as judged by the consistent method followed here, there is no indication of any instability. Furthermore, the inconsistent solutions which did signal instability are likely to be misleading. It is not ruled out that higher order or non-perturbative behavior (inaccessible, by definition, in this formalism, but also by construction, in approximating the action as a truncated perturbative expansion) could make flat space unstable or metastable. But at least for the moment, the issue of stability of flat space is no reason to question general relativity as an approximation to nature, nor to question the present methods of obtaining first order corrections to the field equations of gravity.

V. Summary

Semiclassical and other more systematic approaches (such as $1/N$ expansions) to quantum corrections of gravity depend on the validity of perturbatively expanding the effective action and field equations in powers of $\hbar$. In the case of gravity, the perturbative corrections have the form and dimension of curvature squared terms (though the effective actions may not be entirely expressible in terms consisting of only the metric and curvature), which leads to a higher derivative theory, *i.e.* fourth order in time. Corrections of still higher order, expanded in powers of the Planck length, are expected to be of even higher order in time derivatives. If taken seriously, new solutions to the higher order equations make the system both qualitatively and quantitatively different from the classical case, leading to, among other symptoms, the instability of flat space. Two important features of these apparently new solutions also point the way to the cure. First, most of the new "solutions" are not perturbatively expandable in powers of $\hbar$, in contrast to the effective action and field equations. Secondly, the order of the derivatives increase with increasing perturbative order. These make it plausible that the higher derivatives arise from a perturbative expansion of a non-local system and not from any dynamical considerations. Non-locality is to be expected in the low energy effective action describing gravity in the low curvature limit (as in all effective actions). Still, even if the higher derivative terms arise for reasons other than non-locality, the pseudo-solutions must still be excluded for self-consistency if the effective action examined is a truncated perturbative expansion. This process does not deny the existence of possible non-perturbative behavior of the full theory, but it does acknowledge that such behavior is inaccessible in the perturbative expansion already performed.

The cure is merely to take the perturbative expansion seriously and to exclude all "pseudo-solutions" not perturbatively expandable in $\hbar$. This is necessary for self-consistency: 1) the initial action and field equations are formal perturbative expansions and now the solutions are also formal perturbative expansions; 2) the number of degrees of freedom of the system is fixed and does not depend on the order to which the expansion is taken; 3) the system plus the constraints necessary to exclude the non-perturbative pseudo-solutions is strongly equivalent (in the sense of Dirac constrained systems) to a second order system, and thus has none of the pathologies of unconstrained higher derivative theories. Otherwise, the penalty is spurious solutions to the field equations, unlikely to be related to solutions of the full non-perturbative field equations. A simple model has been provided for which retaining non-perturbative degrees of freedom (as is usually done for semiclassical gravity) gives the wrong answer, and excluding them gives the correct answer. It also demonstrates that that the presence of higher derivative terms in the action and field equations does not automatically require that they will have dynamical consequences as such.

The effect of excluding non-perturbative pseudo-solutions from semiclassical gravity is to restore stability to flat space from quantum corrections, at least perturbatively to first order in $\hbar$. Stability is not proven or guaranteed to all orders or against non-perturbative behavior, but there is no evidence at present to the contrary.

There are other contexts, *e.g.* cosmology, in which semiclassical gravity has been used without excluding non-perturbative pseudo-solutions. Any proposal that depends crucially on the non-perturbative behavior is flawed for the same reasons.

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Appendices

Appendix A

The quantized version of the system presented in equation (2 - 10) can be solved exactly by algebraic methods. We define the new canonical variables

$$\begin{aligned} q_+ &= \frac{1}{\omega}\frac{1}{\sqrt{1-\varepsilon^2\omega^2}}(\varepsilon^2\omega^2\dot{x} - \pi_x) \\ p_+ &= \frac{\omega}{\sqrt{1-\varepsilon^2\omega^2}}(x - \pi_{\dot{x}}) \\ q_- &= \frac{\varepsilon}{\sqrt{1-\varepsilon^2\omega^2}}(\dot{x} - \pi_x) \\ p_- &= \frac{1}{\varepsilon}\frac{1}{\sqrt{1-\varepsilon^2\omega^2}}(\varepsilon^2\omega^2 x - \pi_{\dot{x}}) \end{aligned} \qquad \text{(A - 1)}$$

such that the Hamiltonian is in the form of the difference of 2 harmonic oscillators,

$$H = p_+{}^2 + \omega^2 q_+{}^2 - (p_-{}^2 + \varepsilon^{-2} q_-{}^2) \quad . \qquad \text{(A - 2)}$$

The energy spectrum is then given by

$$E = (n + ½)\omega - (m + ½)\varepsilon^{-1} \qquad \text{for } n, m = 0, 1, 2, \ldots \quad . \qquad \text{(A - 3)}$$

The wave function can then be put in the form

$$\psi_{nm}(\pi_x, \dot{x}) = \chi_{n,\omega}(q_+)\chi_{m,\varepsilon^{-1}}(q_-) \qquad \text{(A - 4)}$$

where $\chi_{n',\omega'}$ is the standard simple harmonic oscillator wave function with energy level n' and frequency ω', and $q_\pm$ are defined above. $\psi_{nm}(x, \dot{x})$ is given by the Fourier transform, *e.g.*

$$\begin{aligned} \psi_{00}(x, \dot{x}) &= \frac{1}{\sqrt{2\pi}}\int d\pi_x \, \mathrm{e}^{ix\pi_x} \, \psi_{00}(\pi_x, \dot{x}) \\ &= \left(\frac{\omega}{\varepsilon\pi^2}\right)^{1/4} \exp\left(-\frac{\omega^2(1-\varepsilon^2\omega^2)x^2 - 4i\varepsilon\omega^2 x\dot{x} - (1-\varepsilon^2\omega^2)\dot{x}^2}{2\varepsilon^{-1}(1+\varepsilon^2\omega^2)}\right) \quad . \end{aligned} \qquad \text{(A - 5)}$$

Appendix B

The rules for reading off the expectation values from the Hamiltonian are simple:

$$\begin{aligned}\left\langle q^{(n)}\right\rangle &= i\hbar\delta \times \text{coefficient of } (p_{q^{(n)}}),\\ \left\langle (q^{(n)})^2\right\rangle &= i\hbar\delta \times \text{coefficient of } \left((p_{q^{(n)}})^2\right), \text{ etc.,}\end{aligned} \qquad \text{(B - 1)}$$

but they only apply when the Hamiltonian/Schroedinger formulation is equivalent to the Feynman Path Integral formulation of quantum mechanics (for more details on when this is true, see, *e.g.* Popov[42]).

To calculate meaningful quantities, take the expectation value of classical expressions (since path integrals are semiclassical approximations at the smallest scale). For example, when calculating the expectation value of velocity, take the transition expectation value of

$$v_{cl} \sim \dot{x} \;\&\; \frac{\Delta x}{\delta} \quad , \qquad \text{(B - 2)}$$

where "&" means "plus terms of order". For the example in (2 - 20), v_{cl} is given exactly by the derivative of (C - 1).

Appendix C

The general solution for the system described by (2 - 20) is

$$
\begin{aligned}
x(t) = \{ \quad & (\dot{x}_i - \dot{x}_f)\sin(\varepsilon^{-1}[T-t]) \\
& +(\varepsilon^{-2}tx_i - \varepsilon^{-2}tx_f - \varepsilon^{-2}Tx_i - \dot{x}_i + \dot{x}_f)\sin(\varepsilon^{-1}T) \\
& +(\dot{x}_i - \dot{x}_f)\sin(\varepsilon^{-1}t) \quad - \quad \varepsilon^{-1}(T\dot{x}_i + x_i - x_f)\cos(\varepsilon^{-1}[T-t]) \\
& +\varepsilon^{-1}(T\dot{x}_f + x_i - x_f)\cos(\varepsilon^{-1}t) \\
& +\varepsilon^{-1}(-t\dot{x}_i - t\dot{x}_f + T\dot{x}_i - x_i - x_f)\cos(\varepsilon^{-1}T) \\
& +\varepsilon^{-1}(t\dot{x}_i + t\dot{x}_f + T\dot{x}_f + x_i + x_f) \qquad \} \\
& \qquad \times \; \varepsilon\{-2\cos(\varepsilon^{-1}t) - \varepsilon^{-1}T\sin(\varepsilon^{-1}t) + 2\}^{-1}
\end{aligned}
\tag{C - 1}
$$

for a particle beginning its motion at x_i at time $t = 0$ and ending at x_f at time $t = T$. From this we can compute the classical action

$$
S = \frac{\varepsilon}{2}E^{-1}[A(\dot{x}_i^2 + \dot{x}_f^2) - 2B\dot{x}_i\dot{x}_f - 2\varepsilon^{-1}C(\dot{x}_i + \dot{x}_f)(x_i - x_f) + \varepsilon^{-2}D(x_i - x_f)^2] \tag{C - 2}
$$

where

$$
\begin{aligned}
A &= -4\varepsilon^{-1}T\cos\varepsilon^{-1}T + 3\varepsilon^{-1}T\cos 2\varepsilon^{-1}T + 4\sin\varepsilon^{-1}T \\
&\qquad +(\varepsilon^{-2}T^2 - 2)\sin 2\varepsilon^{-1}T + \varepsilon^{-1}T \\
B &= 4\varepsilon^{-1}T\cos\varepsilon^{-1}T + \varepsilon^{-1}T\cos 2\varepsilon^{-1}T + 2(\varepsilon^{-2}T^2 + 2)\sin\varepsilon^{-1}T \\
&\qquad -2\sin 2\varepsilon^{-1}T - 5\varepsilon^{-1}T \\
C &= 8\cos\varepsilon^{-1}T - 2\cos 2\varepsilon^{-1}T + 2\varepsilon^{-1}T\sin\varepsilon^{-1}T - \varepsilon^{-1}T\sin 2\varepsilon^{-1}T - 6 \\
D &= \varepsilon^{-1}T\cos 2\varepsilon^{-1}T + 4\sin\varepsilon^{-1}T - 2\sin 2\varepsilon^{-1}T - \varepsilon^{-1}T \\
E &= 16\cos\varepsilon^{-1}T + (\varepsilon^{-2}T^2 - 4)\cos 2\varepsilon^{-1}T + 8\varepsilon^{-1}T\sin\varepsilon^{-1}T \\
&\qquad -4\sin 2\varepsilon^{-1}T - \varepsilon^{-2}T^2 - 12 \quad .
\end{aligned}
\tag{C - 3}
$$

Because the Lagrangian is quadratic, the quantum transition amplitude given by the path integral formulation is exactly

$$K(x_f, \dot{x}_f, t_i + T; x_i, \dot{x}_i, t_i) = F(T)e^{iS} \tag{C - 4}$$

where $F(T)$ can be derived from the classical action alone (see *e.g.* Marinov[43] for details)

$$F(T) = \frac{1}{2\pi i}\left(\frac{-BD - C^2}{E^2}\right)^{1/2} \quad . \tag{C - 5}$$

Schroedinger's equation for this system can be obtained directly from (C - 4) with out the use of canonical formalism or the Hamiltonian.[22]

Appendix D

One may always add any total derivative to a Lagrangian without affecting the equations of motion, but not every Lagrangian had a valid variational formulation associated with it, and without one, the associated quantum mechanical wave function will not fold. A simple example of a Lagrangian without a valid variational formulation is

$$L_\alpha = \tfrac{1}{2}(\dot{x}^2 - \omega^2 x^2) + \alpha \frac{d}{dt}(x\dot{x}) \qquad \text{for} \quad \alpha \neq 0, -1 \tag{D - 1}$$

for which the variational principle is

$$\delta S_\alpha = -\int_1^2 (\ddot{x} + \omega^2 x)\delta x\, dt + (1+\alpha)\dot{x}\delta x\Big|_1^2 + \alpha x \delta\dot{x}\Big|_1^2 \quad . \tag{D - 2}$$

This tells us to fix four boundary conditions for a second order equation, which is an overdetermined system. Any Lagrangian lacking a valid variational formulation can regain it by adding a total derivative. In this case, the most obvious total derivative to add is $-\alpha \frac{d}{dt}(x\dot{x})$. Note that for $\alpha = -1$ there is a valid variational formulation (even though there are also constraints); this is the theory obtained by choosing the canonical momentum as the generalized coordinate. (A valid variational formulation is not needed to put the system into canonical form, it is only necessary for quantum mechanics.)

From equation (2 - 4) and from calculating the $p_{x^{(n)}}$ for our non-local oscillator (expressed as the infinite sum in the second line of (2 - 34), we can see that the model Lagrangian presented above does have a valid variational formulation, once the implied constraints of (2 - 35) have been taken into account. The implied constraints tell us that holding all even derivatives fixed on the boundaries holds x fixed, and holding all odd derivatives fixed holds $\dot{x}$ fixed. Because the $p_{x^{(n)}}$ vanish for odd n,

$$\delta S = \int_1^2 (\text{Equations of Motion}) + (\text{some function})\delta x\Big|_1^2 \qquad \text{(D - 3)}$$

once the constraints are used. It is correct and, in fact, necessary to use the constraints to determine whether or not the system is over determined, as they are just part of the equations of motion (c.f. the case of L_α). In general, adding an arbitrary total derivative to the Lagrangian, if it contains higher derivatives, corresponds to a canonical change in variable, which would destroy the variational formulation.

Appendix E

Here we investigate the effects of the perturbative expansion in examining the stability of a system through its action (which can be effective or exact). Let the full (non-perturbative) action, $\Gamma^{\alpha}[\varphi]$, be a functional of a field φ (not necessarily a scalar) and a function of the parameter α. φ is limited to a particular class of functions, S, *i.e.* $\varphi \in S$. For example, S could be the class of functions held fixed at the boundaries, or of functions and their first derivatives held fixed at the boundaries. We require that that $\Gamma^{\alpha}[\varphi]$ have two properties. First, it must be perturbatively expandable in α

$$\Gamma^{\alpha} \cong \Gamma^{\alpha}_{\text{pert}} = \Gamma_0 + \alpha\Gamma_1 + \ldots \quad \text{where} \quad \Gamma_n \equiv \frac{1}{n!}\frac{\partial^n \Gamma}{\partial \alpha^n}\bigg|_{\alpha=0} \qquad \text{(E - 1)}$$

where "≅" has the specific meaning of "possesses an asymptotic expansion equal to." (Note that, for instance, any term of the non-perturbative form $\exp(-1/\alpha)$ could be added to the left-hand side of (E - 1) and the right-hand side would remain unchanged.) Secondly, there must exist an extremum to the action that is also perturbatively expandable in α.

$$\exists\overline{\varphi}^{\alpha} \text{ such that} \quad \text{a) } \overline{\varphi}^{\alpha} \in S,$$

$$\text{b) } \left.\frac{\delta\Gamma}{\delta\varphi}\right|_{\overline{\varphi}^{\alpha}} = 0, \tag{E - 2}$$

$$\text{and} \quad \text{c) } \overline{\varphi}^{\alpha} \cong \overline{\varphi}^{\alpha}_{\text{pert}} = \overline{\varphi}_0 + \alpha\overline{\varphi}_1 + \ldots \quad .$$

For any action with these properties, the following statements can be proven just from the theory of asymptotic expansions (or strict perturbation theory). First

$$0 = \left.\frac{\delta\Gamma^{\alpha}}{\delta\varphi}\right|_{\overline{\varphi}^{\alpha}} \cong \left.\frac{\delta\Gamma^{\alpha}_{\text{pert}}}{\delta\varphi}\right|_{\overline{\varphi}^{\alpha}_{\text{pert}}} , \tag{E - 3}$$

i.e., the perturbative expansion of the extremal field configuration is also an extremal field configuration of the perturbatively expanded action. This follows trivially from the definition of an asymptotic expansion. The same holds true (with identical proofs) for any number of functional derivatives, and, in particular, for the second functional derivative, which must be positive definite if the system is to be stable.

$$\left.\frac{\delta^2\Gamma^{\alpha}_{\text{pert}}}{\delta^2\varphi}\right|_{\overline{\varphi}^{\alpha}_{\text{pert}}} \cong \left.\frac{\delta^2\Gamma^{\alpha}}{\delta^2\varphi}\right|_{\overline{\varphi}^{\alpha}} \tag{E - 4}$$

A stronger statement than (E - 4) is that

$$\text{If} \quad \left.\frac{\delta^2 \Gamma^{\alpha}_{\text{pert}}}{\delta^2 \varphi}\right|_{\varphi^{\alpha}_{\text{pert}}} = 0, \qquad \text{where} \quad \varphi^{\alpha}_{\text{pert}} = \varphi_0 + \alpha\varphi_1 + \ldots \tag{E - 5}$$

$$\text{then} \quad \varphi^{\alpha}_{\text{pert}} = \overline{\varphi}^{\alpha}_{\text{pert}} \quad ,$$

or, if an extremum of the perturbatively expanded action is itself perturbatively expandable, then it equals the perturbatively expanded extremum of the full action. This is most important because it shows that the perturbatively expandable extrema of the perturbatively expanded action is related in a well defined way to the exact extrema of the full action. No such statement can be made if an extremum of the perturbatively expanded action is not itself a perturbative expansion, and in fact if

$$\left.\frac{\delta^2 \Gamma^{\alpha}_{\text{pert}}}{\delta^2 \varphi}\right|_{\varphi_{\text{test}}} = 0, \qquad \text{where} \quad \varphi_{\text{test}} \neq \varphi_0 + \alpha\varphi_1 + \ldots \tag{E - 6}$$

(a non-perturbative extremum is chosen), then, in general, $\Gamma^{\alpha}[\varphi_{\text{test}}] \not\approx \Gamma^{\alpha}_{\text{pert}}[\varphi_{\text{test}}]$, *i.e.*, the perturbative action, when evaluated at a non-perturbative extremum of itself, is not even approximately equal to the full action evaluated at the same test function. The proof of (E - 5) is straightforward. The left side of the first equation of (E - 5) is an infinite polynomial in α, every term of which must vanish independently (since α is arbitrary). The vanishing of each term determines ϕ_n uniquely (so long as ϕ_0 is unique). Thus the infinite polynomial $\varphi^{\alpha}_{\text{pert}}$ is unique, and must equal $\overline{\varphi}^{\alpha}_{\text{pert}}$ by (E - 2).

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39. See e.g. Julian Schwinger, Phys. Rev. **76**, 749 (1949).

40. There are, of course, non-local systems with perturbatively expandable actions and field equations that do contain non-perturbative solutions. A simple example is the system given by equation (3 - 5) where $\varepsilon \rightarrow \iota\varepsilon$. In this case the non-perturbative pseudo-solutions present in the truncated expansion behave nothing like the non-perturbative solutions to the the full equations of motion, and so still must be excluded for the sake of consistency. One should not expect non-perturbative solutions to be found from a perturbative expansion.

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